Resolution of policy uncertainty and sudden declines in volatility

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ABSTRACT

We introduce downward volatility jumps into a general non-affine modeling framework of the term structure of variance. With variance swaps and S&P 500 returns, we find that downward volatility jumps are associated with a resolution of policy uncertainty, mostly through statements from FOMC meetings and speeches of the Federal Reserve’s chairman. Ignoring such jumps may lead to an incorrect interpretation of the tail events, and hence biased estimates of variance risk premia. On the modeling side, we explore the structural differences and relative goodness-of-fits of factor specifications. We find that log-volatility models with at least one Ornstein–Uhlenbeck factor and double-sided jumps are superior in capturing volatility dynamics and pricing variance swaps, compared to the affine model prevalent in the literature or non-affine specifications without downward jumps.

1. Introduction

Volatility responds to news. It rises dramatically and immediately following the occurrence of unexpected bad events. Moreover, volatility not only jumps upward but also moves downward rapidly. Sudden declines in volatility are sometimes related to stock market rallies stimulated by unexpected good news from economic indicators or earnings announcements. Yet they are also very often triggered by the resolution of policy uncertainty that shifts investors’ sentiment. Recent news headlines bring this fact into the spotlight. In particular, as can be seen from Fig. 1, the VIX dropped 35% on May 10, 2010, as a result of Europe’s emergency loan plan; another 27% on Aug 9, 2011, due to Federal Reserve’s rate statement on keeping interest rates at a record low through mid-2013; and finally 23% on Dec 31, 2012, in anticipation of lawmakers making a deal to avert the “fiscal cliff”.

Despite the size and scope of their bailout is uncertain, the government and Federal Reserve often intervene in the midst of hard times, which effectively provides a put protection on asset prices. Our hypothesis is that many downward volatility jumps are ex-post market reactions to these policy measures, and that they are important sources of risk for volatility traders ex-ante. This type of variance risk should be priced in volatility derivatives, and could be an important part of the total variance risk premia. Therefore, ignoring downward volatility jumps may lead to an incorrect interpretation of the price of tail events. The goal of this paper is to provide a systematic investigation of where downward volatility jumps originate, how they affect asset prices, and whether they are priced risk factors.

These questions invite us to search for appropriate derivatives to investigate the asset pricing implications of volatility shocks. While the S&P 500 options offer a developed battlefield for volatility trading, volatility derivatives have thrived on the demand for volatility hedging and speculation since their inception. The over-the-counter index variance swap contract is one particular example of these popular derivatives. As with most swaps, the fixed leg of variance swaps pays a pre-determined amount at maturity in exchange for the realized variance that the floating leg commits to offer. Despite the path-dependence of realized variance, the payoff structure of variance swaps is appealing for studying the term structure of variance and variance risk premia, as opposed to the exchange-traded VIX derivatives, in that variance swaps directly reflect investors’ expectation on future uncertainty. Moreover,

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a variance swap can be replicated using a portfolio of S&P 500 options, which is very similar to the VIX itself. Therefore it is very sensitive to volatility jumps.

Despite their existence, whether and how these volatility jumps affect asset prices and risk premia remain largely unknown, particularly in the case of the large downward jumps. This is partially due to the absence of derivative pricing models that allow for downward volatility jumps in the mainstream finance literature. Popular affine models such as the square-root volatility models can only incorporate upward jumps in order to ensure the positivity of variance. We incorporate downward volatility jumps and other potentially negative latent factors into a non-affine framework that guarantees the positivity of variance.

With this new and general non-affine framework, we price variance swaps in (quasi) closed form, and identify downward volatility jumps along with two latent volatility factors from 17 years of variance swap data and S&P 500 returns. We find that volatility jumps are often triggered by unexpected macro announcements. In particular, sudden declines in volatility are mostly associated with the resolution of policy uncertainty, such as monetary policy changes that are explicit or implicit from Federal Open Market Committee (FOMC) statements or the speeches of the Federal Reserve’s chairman, as well as fiscal policy decisions and compromises made by Congress.

Among several alternative specifications, we provide compelling evidence in favor of log-volatility models with at least one Ornstein–Uhlenbeck factor. The Ornstein–Uhlenbeck process provides sufficient persistency required for the long-term volatility factor. Our regression analysis shows that latent volatility factors are not only related to excess market returns, but also to liquidity and credit factors, as well as policy news. In particular, policy news are important for the short-term factor, whereas the default risk is paramount for the long-term. In addition, we find that downward volatility jumps are mostly related to the short-term volatility factor, yet have insignificant impacts on the long-term factor.

Unlike prevalent parametric affine models in the literature, our volatility dynamics provide a more flexible specification of variance risk premia. We find that the size of downward volatility jumps is smaller under the risk neutral measure, suggesting that market participants are pessimistic about the scale of the intervention ex-ante. In addition, our estimates conform with the existing model-free estimates that the total variance risk premia are negative most of the time, yet they tend to be insignificant or even positive at the inception of crises. This finding is a puzzle as it is in conflict with a representative agent model widely adopted in the literature.

There is a growing amount of theoretical and empirical work relating political uncertainty to asset pricing. In particular, Pastor and Veronesi (2013) relate the stock market risk premia, volatility, and correlation to the policy uncertainty index constructed by Baker et al. (2013) which is based on the frequency of newspaper references to economic policy uncertainty and other indicators. The regression results of Pastor and Veronesi (2013) agree with all the predictions of their learning model, see also Pastor and Veronesi (2012) for another related model of government policy choice. Boutchkova et al. (2012) investigate how local and global political risks affect industry return volatility. Kelly et al. (2016) find evidence for government guarantee premia by examining the basket-index spread from out-of-the-money put options. Bernanke and Kuttner (2005) study stock market reactions to Federal Reserve policy and find that the effects of unanticipated monetary policy actions on expected excess returns account for the largest part of the responses of stock prices. In turn, Beber and Brandt (2009) investigate the link between ex-ante macroeconomic uncertainty and ex-post uncertainty resolution in financial markets, using the prices of some options whose underlying is

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4 While many macro announcements are pre-scheduled, their impact remains unexpected. As a result, the literature resorts to Poisson processes for modeling jumps, with notable distinctions by Maheu and McCurdy (2004), Piazzesi (2005), Dubinsky and Johannes (2006) and Beber and Brandt (2009).
the release of non-farm payroll. They find that higher ex-ante uncertainty leads to a larger reduction in volatility along with a greater increase in trading activity after the news release. While these studies have shed light on the link between political uncertainty and risk in equity markets, we further point out that sudden decreases in volatility are particularly related to the resolution of monetary policy uncertainty, ignoring which may lead to misleading interpretations of the tail events and biased variance risk premia.

Our empirical findings on volatility jumps are also relevant to the large literature that investigates the unique role of jumps in asset pricing, which dates back as early as Merton (1976), who introduces jumps to model index returns. Since the seminal work by Duffie et al. (2000), positive volatility jumps, exponentially distributed, have been constantly added to model index volatility dynamics, so that volatility can jump upward but revert back to the mean slowly. Eraker (2003), in particular, point out the role played by such volatility jumps and compare them to the role of jumps in returns. However, models in the literature that discuss the existence and necessity of downward volatility jumps are rare. An exception is Andersen et al. (2015a), who document downward volatility jumps using the constructed intraday corridor implied volatilities. Also, Todorov and Tauchen (2011) investigate the activity of volatility jumps using high-frequency historical returns of the VIX. Their specification allows downward volatility jumps. In contrast, we focus on the asset pricing implications of volatility jumps, which require modeling the risk-neutral and the objective dynamics jointly. Recently, Chernov et al. (2017) have discussed the impact of jumps on exchange rates and the impact of positive jumps on their variances, and they relate these to macroeconomic and political news. They find few positive jumps in variance that respond to such news. We also find positive jumps less responsive to political news, unlike negative jumps.

Previous work in the literature on variance swaps is mostly based on fully specified parametric models using both variance swaps and index values. Egloff et al. (2010) and Amengual (2008) find that single-factor volatility models are incapable of fitting the term structure of variance swap rates. They therefore suggest applying models with two-volatility factors to investigate the term structure of variance. None of their models have volatility jumps. Aït-Sahalia et al. (2014) propose a similar affine model with positive volatility jumps to estimate the liquidity and variance risk premia. They focus on the component of variance risk premia due to price jumps. Li and Zinna (2017) further add a self-exciting jump factor to the same affine model. Fusari and Gonzalez-Perez (2012) consider a log-affine model with two Ornstein–Uhlenbeck factors but without volatility jumps, in addition to an affine model. Similarly, Carr et al. (2012) focus on the pricing and hedging of variance swaps and volatility derivatives using time-changed Lévy processes. Filipovic et al. (2016) independently propose a class of quadratic models without volatility jumps. All the aforementioned continuous-time models are nested within our framework. Recently, Dew-Becker et al. (2017) propose to investigate structural economic models using variance swaps. Their affine models are cast in discrete-time and only positive volatility jumps are allowed. While these two-factor volatility models without volatility jumps have been shown to yield accurate variance swap prices, our empirical results suggest that their dynamics under the objective measure are likely misspecified.

To study variance risk premia, many papers adopt alternative nonparametric techniques. Among others, Bakshi and Kapadia (2003) estimate variance risk premia using delta-hedged gains of S&P 500 options, whereas Carr and Wu (2008) study variance risk premia using synthetic variance swaps for individual firms and indexes. The synthetic variance swap price thereafter becomes a popular proxy of the risk-neutral conditional variance. To measure the conditional variance under the objective measure, Bollerslev et al. (2011) suggest the use of high-frequency five-minute-based realized volatilities, see also Zhou (2009). Although realized volatilities are model-free estimates, estimating the objective conditional expectation requires a parametric forecasting model. In this regard, Bekaert and Hoerova (2014) evaluate a plethora of state-of-the-art forecasting models to produce an accurate measure of the conditional variance, and point out that a non-linear model may be better equipped to capture the behavior of conditional variance and variance risk premia in severe crises. We specify and estimate non-affine dynamic models for the index return and its volatility, hence we are able to address their conjecture. Moreover, our full-fledged and unified model facilitates the joint statistical inference on conditional variances under both measures.

Our paper is also related to the specification of models that can capture index volatility dynamics, one of the central themes in empirical option pricing and financial econometrics. This strand of literature investigates the volatility dynamics through the lens of S&P 500 options, see, e.g., Bakshi et al. (1997), Bates (2000), Pan (2002), Eraker (2004), and Broadie et al. (2007) for examples of affine jump diffusion models with stochastic volatility driven by one square-root factor. Recent findings by Christoffersen et al. (2009) and Bates (2012) also suggest that models with two square-root factors are essential for capturing the term structure of variance. Andersen et al. (2015b) argue for more factors in order to capture the time-varying skewness of the implied volatility. All these papers focus on affine volatility models, in particular the square-root models. Nevertheless, ample evidence from historical time series of stock returns supports log-volatility models, including discrete-time ones by French et al. (1987), Schwert (1990), Nelson (1991), as well as continuous-time models, potentially with jumps or even comprised purely of jumps, e.g., Barndorff-Nielsen and Shephard (2001), Chernov et al. (2003), and Todorov and Tauchen (2011). Indeed, log-volatility models naturally allow downward volatility jumps since they always guarantee the positivity of variance. Plus, log-volatility models allow Ornstein–Uhlenbeck factors, which are not restricted by a similar Feller’s condition for square-root processes. Therefore, they allow for more persistent volatility factors. Empirically, Feller’s condition is often binding for the risk neutral dynamics, even for models with multiple volatility factors, see, e.g., Song and Xiu (2016). The drawback of these log-volatility models lies in their lack of tractability for option pricing. As mentioned earlier, instead of relying on options we resort to variance swaps and derive a (quasi) closed-form pricing formula, using which we can investigate the pricing implications of log-volatility models. There are a couple of papers in the empirical option pricing literature, though, which investigate non-affine risk neutral dynamic models using simulation methods, e.g., Christoffersen et al. (2006) and Durham (2013). However, conducting statistical inference on top of simulated prices is computationally intensive.

This paper is organized as follows. Section 2 presents our framework for variance swap modeling. Section 3 discusses the statistical inference and provides simulation evidence, followed by empirical results in Section 4. Section 5 concludes the paper. The supplemental appendix includes mathematical proofs, technical details, as well as additional tables and figures.

2. Variance swap modeling

A variance swap contract is an over-the-counter derivative in which the contract holder pays at maturity $t + \tau$ a fixed amount (variance swap rate) for the realized variance:

$$\frac{1}{\tau} \sum_{i=1}^{\lfloor \tau/\Delta \rfloor} \left( Y_{t+i\Delta} - Y_{t+(i-1)\Delta} \right)^2.$$
where \( Y \) is the log-price of the underlying index, i.e. S&P 500 index. By entering long positions in such contracts, investors can hedge against high realized variance. Thus, the differences between the expectation of variance and the swap price, i.e., the variance risk premia investors earn, are typically negative, see, e.g., Bollerslev et al. (2009) and Drechsler (2013).

Variance swap trading has grown rapidly since the aftermath of the LTCM turmoil in late 1990s. For investors using medium- or low-frequency trading strategies, these over-the-counter contracts are more favorable than S&P 500 options for the purpose of volatility trading, since investors can express their views on volatility without having to do labor-intensive delta hedging.

We start by proposing a full-fledged multi-factor non-affine volatility model for which we provide a general pricing formula of variance swaps in Section 2.1. Section 2.2 specifies the risk premia and the dynamics under the objective measure. Section 2.3 discusses canonical forms and identification. We then provide examples of two-factor volatility models in Section 2.4, which we use in the empirical study.

2.1. Risk neutral modeling and pricing

As is well known, realized variance converges (in probability) to the quadratic variation of \( Y \), i.e. \( [Y, Y]_{t\rightarrow\tau} \), and modeling the quadratic variation is a common practice that facilitates the variance swap pricing.\(^5\) Since there is no money changing hands at the initiation of the trade, i.e., time \( t \), the variance swap rate, under some risk neutral measure \( Q \), is given by:

\[
P(t, \tau) = 100 \times \frac{1}{\tau} \mathbb{E}_Q^\tau \left\{ \left[ Y, Y \right]_{t\rightarrow\tau} \right\} = 100 \times \frac{1}{\tau} \mathbb{E}_Q^\tau \left\{ \int_t^{t+\tau} \sigma^2_s ds + \int_t^{t+\tau} \int R \nu_\sigma^0(df)ds \right\},
\]

where the calculation is based on the usual specification of the risk neutral dynamics of \( Y \):

\[
dY_t = \mu_t dt + \sigma_t dB_t^Q + dJ_t^Q,
\]

where \( B_t^Q \) is a Brownian motion, \( J_t^Q \) is a compensated jump process with compensator \( \nu_\sigma^0(\cdot) \), \( \sigma_t \) is a volatility process, and \( \mu_t \) is the drift determined by the no-arbitrage condition.

2.1.1. Variance dynamics

We model the variance as certain non-affine function of some factors summarized in \( X \):

\[
\sigma_t^2 = \Pi_0 + \Pi_1^\top X_t + \Pi_2 X_t \exp \left\{ \Pi_3 + \Pi_4^\top X_t \right\},
\]

where \( \Pi_0 - \frac{1}{2} (\Pi_1^\top (\Pi_2^{-1}) \Pi_1) \geq 0 \), which warrants a positive variance. This model augments the exponentially affine specification by a quadratic component, hence nesting the common affine cases when \( X \) only takes positive values, as well as the quadratic variance swap model by Filippovic et al. (2016). In fact, the specification of variance in (2) can be generalized to the so-called tempered distributions.\(^6\) We choose quadratic and exponential functions because they are simple and nest the common specification of variance dynamics in the literature.

To ensure the tractability of this general non-affine class of models, we assume that the underlying \( N \)-dimensional factor \( X \) follows a multivariate affine process, similar to the affine term structure model discussed in Dai and Singleton (2000), but allowing for jumps, e.g., as in Duffie et al. (2000) and Chen and Joslin (2012).\(^7\) We write the risk neutral model of \( X_t \) as:

\[
dx_t = (A^Q X_t) dt + \Sigma \sqrt{\Sigma} dW_t^Q + dZ_t^Q,
\]

where \( W_t^Q \) is an \( N \)-dimensional standard Brownian motion, and \( \Sigma \) is a diagonal matrix in \( \mathbb{R}^{N \times N} \), with \( [\Sigma]_{ii} = \alpha_1 + \beta_i^2 X_t \), and \( Z_t^Q \) is another compensated jump process. While there is no need to introduce correlations among \( W_t^Q \) because of \( \Sigma \), we impose a correlation between \( B_t^Q \) and each element of \( W_t^Q \) to incorporate the so-called “leverage effect”.

While the factor \( X \) is restricted within the affine class, the volatility dynamics is non-affine, which leads to several differences compared with the usual term structure models. For example, even when \( X \) is a homoscedastic Gaussian factor, \( \sigma_t^2 \) is heteroscedastic and non-Gaussian, as is obvious from Itô’s lemma. Moreover, the volatility of volatility is another (non-affine) function of \( X \).\(^8\)

2.1.2. Jumps

To specify jumps in both \( Y \) and \( X \), there are trade-offs that must be considered. First, Poisson type jumps are our preferred choice for modeling daily data, as Lévy type jumps are difficult to identify and disentangle from Brownian shocks generated by stochastic volatility at a daily frequency.\(^9\) Second, if the intensities of Poisson jumps are independent for \( Y \) and \( X \), then there would be no co-jumps of \( Y \) and \( X \) almost surely, which conflicts with the data, see, e.g., Jacob and Todorov (2010). Third, there are many pre-scheduled macro announcements, FOMC meetings, and speeches by Federal Reserve Chairmen, which potentially cause jumps on the market. Hence it may be reasonable to model jumps with deterministic timing, see, e.g., Piazzesi (2005), Maheu and McCurdy (2004), Dubinsky and Johannes (2006), and Beber and Brandt (2009). However, there are many days in our sample with at least one such event, and jumps are not always present. From our empirical analysis below, whether a jump occurs or not on a scheduled event depends on the extent of the news surprise, i.e., the content of the announcement. Most of these events do not lead to jumps. Fourth, from the perspective of risk premia estimation, the major difference between deterministic timing and the random arrival of jumps lies in the risk premia associated with the intensity of the jumps — there are no risk premia for the deterministicarrival of jumps, which may not be the case for the random arrival. Hence, for the sake of parsimony, we conform with the common practice in the literature, e.g., Pan (2002), and model jumps in \( Y \) and \( X \) as compound Poisson processes driven by the same intensity, with no price of risk associated with the jump intensity. Therefore, we write:

\[
(j_t^Q, Z_t^Q) = \int_0^t \int R (j, z) (\mathbb{N}(ds, dj, dz) - \nu_\sigma^0(dj, dz)ds),
\]

where \( \mathbb{N} \) is the Poisson random measure, \( \nu_\sigma^0(dj, dz) \) denotes its compensator, and the Poisson jump intensity is given by \( l_0 + l_1^Q X_t \) with \( l_0 \in \mathbb{R}_+ \) and \( l_1 \in \mathbb{R}_+^N \), \( l_1 \) only has non-zero and positive loadings on positive factors in \( X \).

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\(^5\) Practitioners price variance swaps using a replicating portfolio of options, which relies on the same quadratic variation approximation, see Bossu et al. (2005), hence the discretization error can be ignored, e.g., Jiang and Tian (2007).

\(^6\) Tempered distribution refers to the distribution of functions in the Schwarz space, a linear space of functions all of whose derivatives are rapidly decreasing, see Stein and Shakarchi (2003). Fourier transform is well-defined for a tempered distribution, see Kanwal (2004). The term “distribution” here should not be confused with the “distribution” in statistics.

\(^7\) As shown in Cheridito et al. (2010), the canonical forms in Dai and Singleton (2000) are not exhaustive, unless \( m \leq 1 \) or \( m \geq N - 1 \). While Duffie et al. (2003) and Joslin (2016) propose more general affine processes, we adopt the model used by Dai and Singleton (2000) for its popularity and simplicity.

\(^8\) The volatility of volatility, \( \sqrt{\sigma^2_t} / \sigma_t \), can be calculated by Itô’s lemma, where \( [\cdot, \cdot] \) denotes the continuous component of the quadratic variation. Since \( \sigma_t^2 \) is a non-linear function of \( X_t \), \( \sqrt{\sigma^2_t} / \sigma_t \) is in general a non-linear function of \( X_t \).

\(^9\) There is a large literature on jump detection with intraday data, see e.g. Huang and Tauchen (2005), Barndorff-Nielsen and Shephard (2006), Lee and Mykland (2008), Jiang and Oomen (2008), Jacob and Todorov (2009), and Aït-Sahalia and Jacob (2009).
2.1.3. Variance swaps valuation

We now derive the variance swap rate under the proposed model:

**Proposition 1.** Suppose the risk neutral dynamics follow (1), (2), (3) and (4). The variance swap rate is given in (quasi) closed form by:

\[
P(t, \tau, X_t) = \frac{100}{t} \left\{ \int_t^{t+\tau} \left[ \Pi_0^Q - \frac{\partial}{\partial u_1} \Psi(s, t, u, X_t) \right] du + \int_t^{t+\tau} \left[ \nabla_u \Pi_0^Q \nabla_v \Psi(s, t, u, X_t) \right] ds \right\},
\]

where \( \nabla_u = \left( \partial/\partial u_1, \ldots, \partial/\partial u_N \right)^T \) is a derivative operator, \( \Psi(s, t, u, X_t) = \mathbb{E}[\Pi(s, t, \Psi, X_t)] \) is a derivative operator, \( \Psi(s, t, u, X_t) = \mathbb{E}[\Pi(s, t, \Psi, X_t)] \) is a derivative operator, \( \Psi(s, t, u, X_t) = \mathbb{E}[\Pi(s, t, \Psi, X_t)] \) is a derivative operator, \( \Psi(s, t, u, X_t) = \mathbb{E}[\Pi(s, t, \Psi, X_t)] \) is a derivative operator, \( \Psi(s, t, u, X_t) = \mathbb{E}[\Pi(s, t, \Psi, X_t)] \) is a derivative operator, \( \Psi(s, t, u, X_t) = \mathbb{E}[\Pi(s, t, \Psi, X_t)] \) is a derivative operator, \( \Psi(s, t, u, X_t) = \mathbb{E}[\Pi(s, t, \Psi, X_t)] \) is a derivative operator, \( \Psi(s, t, u, X_t) = \mathbb{E}[\Pi(s, t, \Psi, X_t)] \) is a derivative operator, and \( \Pi_0^Q = \Pi_0 + l_0 \int_0^t \Pi^Q(dj) \). The state variable in the diffusion matrix are non-negative. For each \( m \), we partition \( X^T = (X_{m+1}^T, X_{N-m+1}^T) \). We present the extended canonical forms below. For reasons of space, these canonical forms do not allow pure jump factors. We provide extended canonical forms with pure jump factors in Supplemental Appendix C.

**Definition 1.** The extended canonical representation takes a special form of Eq. (3), where for \( m > 0 \),

\[
K^Q = \begin{pmatrix}
K_{m \times m} & 0_{m \times (N-m)} \\
0_{(N-m) \times m} & K_{(N-m) \times (N-m)}
\end{pmatrix},
\]

and \( \bar{K}^Q \) is either the upper or lower triangle for \( m = 0 \). In addition,

\[
A^Q = \begin{pmatrix}
A_{m \times m} \\
0_{(N-m) \times m}
\end{pmatrix}, \quad \Sigma = I_{N \times N}, \quad \alpha = \begin{pmatrix}
0_{m \times 1} \\
I_{(N-m) \times m}
\end{pmatrix},
\]

with admissibility restrictions (existence conditions) such that for \( 1 \leq i \neq k \leq m \) and \( 1 \leq j \leq N \),

\[
b_{ij} \geq 0, \quad l_{i,j} \geq 0, \quad b_0 \geq 0, \quad \bar{v}^Q(R^m \times R^{N-m}) = 0.
\]

Our specification of jumps leads to the following boundary nonattainment condition:

\[
K^Q_{ii,m} \geq 0, \quad A^Q_{i,j} - b_0 \int_{\mathbb{R}} z_i \bar{v}^Q(dx) \geq \frac{1}{2}, \quad 1 \leq i \neq k \leq m,
\]

as well as the stationarity condition:

\[
\text{Re}(\text{Eigen}(\bar{K}^Q)) < 0,
\]

where, using Diag as an operator that maps a vector to a diagonal matrix,

\[
\bar{K}^Q = \begin{pmatrix}
K^Q_{m \times m} & \text{Diag} \left( l_{i,j}, \int_{\mathbb{R}} z_i \bar{v}^Q(dx) \right)_{1 \leq i \leq m} & 0_{m \times (N-m)} \\
0_{(N-m) \times m} & K^Q_{(N-m) \times (N-m)}
\end{pmatrix}.
\]

These conditions are similar to those in Ait-Sahalia and Kimmel (2010) when jumps are absent and the entire model is affine.

Similar to Dai and Singleton (2000), by the proof in Supplemental Appendix B we have:

**Proposition 2.** For any process that satisfies (1), (2), (3) and (4), there exists a unique canonical representation that is observationally equivalent to it.
Therefore, the canonical representation $A_{a}(N)$ is not only admissible, but it is also maximal in the sense that for each $m$ we impose minimal known sufficient conditions for admissibility and minimal normalizations for econometric identification.

2.4. Examples of two-factor volatility models

Modeling volatility as a two-factor process is an established approach from the literature. Engle and Rangel (2008) decompose volatility shocks into their short-term and long-term components, and relate the long-term component to business cycles in a comprehensive international setting. Adrian and Rosenberg (2008) also decompose equity volatility into similar components, and in addition relate the short-term component to market skewness risk with a cross-section of equity returns. Corradi et al. (2013) directly model the market volatility as a combination of business cycle factors and one additional latent factor, and find that their macro-factors explain the majority of volatility fluctuations. Christoffersen et al. (2009) also find a two-factor volatility structure necessary to model S&P 500 options.

Specifically, we write the risk-neutral dynamics of $X$, a special case of (3), as

$$
\frac{dX_{t}}{dX_{t-1}} = \left( \begin{array}{c}
\lambda_{11}^{Q} + \lambda_{12}^{Q}
\end{array} \right) dt + \left( \begin{array}{c}
\sqrt{\alpha_{1} + \beta_{11}X_{t-1} + \beta_{12}X_{t-1}^{2}}
\end{array} \right) \left( \begin{array}{c}
\Sigma_{11}^{Q} + \Sigma_{12}^{Q}
\end{array} \right) dW_{t}^{P} + \left( \begin{array}{c}
\sqrt{\alpha_{2} + \beta_{21}X_{t-1} + \beta_{22}X_{t-1}^{2}}
\end{array} \right) \left( \begin{array}{c}
\Sigma_{21}^{Q} + \Sigma_{22}^{Q}
\end{array} \right) dW_{t}^{Q},
$$

(7)

where jumps compound Poisson processes with independent jump sizes following double exponential distributions:

$$
\text{size of } Z_{t}^{Q_{1}} \sim \begin{cases}
\text{exp}(\beta_{1+}) : q_{1}, \\
-\exp(\beta_{1-}) : 1 - q_{1},
\end{cases}
$$

$$
\text{size of } Z_{t}^{Q_{2}} \sim \begin{cases}
\text{exp}(\beta_{2+}) : q_{2}, \\
-\exp(\beta_{2-}) : 1 - q_{2}.
\end{cases}
$$

Their intensity is specified as $l_{0} + l_{1}X_{t-1} + l_{2}X_{t-1}^{2}$. In the canonical forms, the specification of jump size distribution is flexible. For parsimony, we employ a simple double-exponential distribution so as to allow for downward jumps as well as the asymmetry in the size of upward and downward jumps. Note that a double-exponential distribution is a natural extension of the exponential distribution typically used in the literature.

Eq. (7) nests the three canonical forms we consider in Sections 3 and 4: $A_{0}(2)$, $A_{1}(2)$, and $A_{2}(2)$, with each allowing for a two-factor structure, and with the first two allowing for negative volatility jumps. We spell out the details of these models in Supplemental Appendix D.

For comparison purposes, in addition to these maximal non-affine models, we also fit three special cases in our empirical study, including the special cases of $A_{0}(2)$ and $A_{1}(2)$ without negative jumps (i.e., $q_{1} = 1$ and $q_{2} = 1$), as well as the affine special case of $A_{2}(2)$ (i.e., $l_{2} = 0, l_{1} = 0, l_{4} = 0$). We denote them $A_{0}^{+}(2)$, $A_{1}^{+}(2)$, and $A_{2}^{+}(2)$, respectively. $A_{2}^{+}(2)$ is a widely used model in the literature, see, e.g., Englof et al. (2010) and Ait-Sahalia et al. (2014).

While we specify different models for volatility, they share the same return dynamics (1). We assume that the size of $J_{t}$ follows a Gaussian distribution with mean $\mu_{t}$ and variance $\sigma_{t}^{2}$. As a result, the compensator in $dJ_{t}$ is $\mu_{t} (l_{0} + l_{1}X_{t}) dt$ and the drift $\mu_{t}$ can be written down explicitly as:

$$
\mu_{t} = r_{t} - d_{t} - \frac{1}{2} \sigma_{t}^{2} - \left( e^{\mu_{t}^{2} + \frac{1}{2} \sigma_{t}^{2}} - 1 - \mu_{t}^{2} \right) (l_{0} + l_{1}X_{t}).
$$

Since the interest rate $r_{t}$ and dividend $d_{t}$ do not affect variance swap prices, their risk neutral dynamics are not identified, hence they are left unspecified.

As the payoff of variance swaps depends on the underlying index $Y_{t}$ only through its risk neutral quadratic variation, variance swaps contain much less information about the dynamics of $Y_{t}$, compared to European options. As a result, with variance swaps, we can only identify the risk neutral jumps of $Y$ up to the expected quadratic variation. For this reason, we impose no market price of risk on the variance of jump sizes in $Y$, so that this parameter can be identified from the $P$-measure dynamics using the S&P 500 index. The mean of jump sizes in $Y$ absorbs all the risk premia of the price jumps, which can be identified from the expected quadratic variation under the $Q$-measure. This assumption is also imposed by, e.g., Ait-Sahalia et al. (2014).

Finally, the $P$-measure dynamics with two volatility factors can be written explicitly as:

$$
\frac{dX_{t}}{dX_{t-1}} = \left( \begin{array}{c}
\lambda_{11}^{Q} + \lambda_{12}^{Q}
\end{array} \right) dt + \left( \begin{array}{c}
\sqrt{\alpha_{1} + \beta_{11}X_{t-1} + \beta_{12}X_{t-1}^{2}}
\end{array} \right) \left( \begin{array}{c}
\Sigma_{11}^{Q} + \Sigma_{12}^{Q}
\end{array} \right) dW_{t}^{P} + \left( \begin{array}{c}
\sqrt{\alpha_{2} + \beta_{21}X_{t-1} + \beta_{22}X_{t-1}^{2}}
\end{array} \right) \left( \begin{array}{c}
\Sigma_{21}^{Q} + \Sigma_{22}^{Q}
\end{array} \right) dW_{t}^{Q},
$$

(8)

with double-exponentially distributed jumps but different parameters from those under the $Q$-measure. For each canonical form under $Q$, we adopt a $P$-model within the same category. More specifically, for each of $A_{0}(2)$, $A_{1}(2)$, and $A_{2}(2)$, we adopt the same constraints on $K^{P}$ as that on $K^{Q}$, but leave $A^{P}$ unconstrained to obtain more flexibility for the market prices of risk, which are implicitly defined as the differences between $P$ and $Q$. This does not affect the identification of $A^{P}$ using $X_{t}$. We set $\Pi_{t}$ to be the same under the two measures, so that $\Pi_{t}$ is identified from the variance swap prices alone. The $P$-dynamics of $A_{0}^{+}(2)$, $A_{1}^{+}(2)$, and $A_{2}^{+}(2)$ are in turn determined as special cases.

In the dynamics of returns, (5), we assume that the size of jumps under $P$ is Gaussian with mean $\mu_{t}$ and variance $\sigma_{t}^{2}$. The intensity is the same under $P$ and $Q$, i.e., $l_{0} + l_{1}X_{t}$. We do not, however, specify $\mu_{t}$, as it is poorly estimated from variance swaps or even from options, and the focus of this paper is not on equity risk premia. Therefore, in our estimation, we follow Eraker et al. (2003) and Eraker (2004) by treating $\mu_{2}$ as a constant. We also confirm in simulations below that such misspecification does not have any noticeable impact on the inference of the remaining parameters.

Overall, the total numbers of parameters equal to 34 for the $A_{0}(2)$, $A_{1}(2)$, and $A_{2}(2)$ models, and 28, 31, and 28 for the $A_{0}^{+}(2)$, $A_{1}^{+}(2)$, and $A_{2}^{+}(2)$ models, respectively. A full list of parameters is available from the first columns of Tables 2–5.

3. Likelihood inference

Our estimation strategy relies on observations of the joint time-series of the underlying S&P 500 index and several variance swap rates with different maturities. However, as is common in many financial models with jump diffusions, likelihood functions are not available. In addition, our state variables are latent and non-Gaussian. Moreover, as discussed in Section 4.1, our panel of data is unbalanced. Therefore, we resort to Markov Chain Monte Carlo (MCMC) methods, see, e.g., Johannes and Polson (2010) for a detailed survey.

12 Once we specify a $Q$-canonical form, $P$-dynamics is partially determined by the Girsanov Theorem. Therefore, using the same canonical form under $P$ imposes restrictions on risk premia implicitly.
3.1. Posterior simulator

We assume that there are observations available on S&P 500 returns and k different variance swap rates and that these observations are recorded at a daily frequency $\Delta = 1/252$, and that the total number of time periods under consideration is $T$. Let $Y_t$ denote the $T \times 1$ vector of S&P 500 prices, and $P$ denote the $T \times k$ panel of variance swap rates.

For convenience, we introduce $V$ and $\theta$ to summarize latent variables and parameters. Typically, $V$ will contain the latent factors in $X$ of the model as well as the remaining latent variables such as jump sizes (denoted by $j_i$ and $z_i$) and jump times (denoted by $u_i$), even though they do not enter into the pricing formula. As for $\theta$, we split it into $(\theta_M; \theta_{\Pi}, \theta_{\Pi}^g; \theta_{\Pi}^q)$. $\theta_M = (\Lambda^g, \text{vec}(K^g), \alpha_i, \beta_i)^{n_1 \times 1}, \beta_i^{n_2 \times 1}$, which contains the parameters determining the dynamics of the latent factors under the risk neutral measure, with $\theta^g_2$ denoting the parameters governing the jump processes; $\theta_{\Pi} = (\Pi_0, \text{vec}(\Pi_1), \text{vec}(\Pi_2), \Pi_3, \text{vec}(\Pi_4))$ includes the parameters defining $\sigma^g_i$; $\theta_{\Pi}^g = (\Lambda^g, \text{vec}(K^g), \rho_1, \rho_2, \beta^g_i)$ summarizes the remaining $\Pi$-measure parameters; and finally, given that there are more derivatives than sources of uncertainty in the theoretical model we allow pricing errors to avoid stochastic singularity. The pricing errors are also economically important in that they capture the remaining factors that are not captured by our pricing model, such as the illiquidity factor or the counterparty risk factor. Specifically, we assume additive pricing errors $\epsilon^g_i$ associated with time $i\Delta$ for variance swap with maturity $j$, so that the observed price satisfies

$$P_i = P(i\Delta, \tau_j, X_{i\Delta}; \Theta_{M}, \Theta_{\Pi}) + \epsilon^g_i,$$

with $\epsilon^g_i$ following a zero-mean Gaussian distribution with variance $\sigma^g_i$, and $\epsilon^g_i$ is independent of $\epsilon^g_j$ for $h \neq j$ and across time. Therefore, the pricing errors are heteroscedastic in the cross-section. $\sigma^g_i$’s are stacked in $\theta_{\Pi}$. Notice that all prices except the S&P 500 index are assumed to be observed with error in our framework and that there is no need to assume that certain combinations of variance swap prices are perfectly observed, unlike the common practice in the literature on the term structure of interest rate.

The purpose of MCMC sampling is to obtain a sample of parameters $\theta$ and latent variables $V$ from their posterior density. Specifically, for a given model $M$, the posterior distribution is given by

$$p(V, \Theta|Y, P, M) \propto L(Y, P|V, \Theta, M) \cdot h(V|\Theta, M) \cdot p(\theta|M) \cdot (9)$$

where $L(Y, P|V, \Theta, M)$ denotes the likelihood function, $h(V|\Theta, M)$ is the density for the latent variables, and $p(\theta|M)$ is the prior density over the parameter vector $\theta$.

We use a Gibbs sampling procedure to estimate these models. In essence, this amounts to reducing a complex problem, i.e., sampling from the joint posterior distribution, into a sequence of tractable ones, i.e., sampling from conditional distributions for a subset of the parameters conditional on all the other parameters, for which the literature already provides a solution. The Gibbs sampling procedure involves sampling sequentially from several blocks:

- **Latent factors**: $p(V_{M|Z}^g|V_{Z|Z}^g, V_{Z|Z}^{g-1}, Z_{Z|Z}^{g-1}, \tau_i, L_i, Z_i, P, Z_i, P)$
- **Q-measure parameters**: $p(\theta_{M|Y}^q|V_{M|Z}^g, \epsilon_{g|Y}^{s-1}, Y, P)$
- **P-measure parameters**: $p(\theta_{\Pi}^P|V_{M|Z}^g, \epsilon_{g|Y}^{s-1}, P)$
- **Pricing equation parameters**: $p(\theta_{\Pi}^g|V_{M|Z}^g, \epsilon_{g|Y}^{s-1}, P)$
- **Pricing error variances**: $p(\theta_{\Pi}^g|V_{M|Z}^g, \epsilon_{g|Y}^{s-1}, \epsilon_{g|Y}^{s-1}, P)$
- **Jump processes**: $p(j_{i\Delta}|Z_i, \theta_{g|Y}^{s-1}, N_{i\Delta}^e|V_{M|Z}^g, \epsilon_{g|Y}^{s-1}, \theta_{g|Y}^{s-1}, Y)$

Supplemental Appendix E contains a detailed description of how we sample the relevant quantities for each of the sampling blocks.

The empirical results shown later are based on 2,000,000 draws. The first 1,200,000 draws are disregarded as burn-in and of the remaining 800,000, one every 80 draws is retained. We also run 4 additional chains for each model to check the convergence of the estimation.

3.2. Choice of priors

In Table 1 we summarize the priors we use by reporting their type of distribution, mean, standard deviation and 95% highest density region for the different elements of $\theta$. The priors for most of the elements of $\theta_M$ and $\theta_{\Pi}$ are uninformative: Gaussian priors with zero mean and large standard deviations. We choose conjugate Gaussian priors for those parameters in $\theta_{\Pi}$ that can be sampled directly from their conditional posterior. Moreover, we use the same mean and variance for these parameters under $P$- and Q-measures to avoid imposing prior information about the sign and magnitude of the risk premia. For convenience, we do not impose the stationarity or boundary nonattainment conditions explicitly through priors. We nonetheless impose admissibility conditions, e.g., sign or range restrictions on parameters that appear in the diffusion, jump sizes, and jump intensities. To disentangle Brownian increments from jumps, i.e., to reflect the nature of jumps as large and infrequent changes in returns and volatility factors, we set slightly more informative priors for jump size parameters such that they place small probability in small jumps, as in Eraker et al. (2003). Finally, the choice of an Inverse Gamma prior for $\sigma^2$ in $\theta_E$ allows us to sample directly from the conditional posterior of that parameter.

3.3. Monte Carlo simulations

In this subsection we discuss the simulation results for $\lambda_0(2), \lambda_1(2)$ and $\lambda_2(2)$. We omit the results for their special cases $\lambda_0(2)$, $\lambda_1(2)$, and $\lambda_2(2)$, as they are similar. For each model, we simulate 20 samples that share the same length and characteristics with the real unbalanced variance swap panel. We also use parameter values close to their estimates.

Tables A.1 and A.2 report the true parameter values, as well as the bias, standard deviation, and 95% high probability regions of the posteriors based on 40,000 draws, from which one every 100 is retained. The parameters in $\theta_M$ and $\theta_{\Pi}$ are precisely estimated. In contrast, the posterior distributions of the drift parameters in $\theta_E$ have a large dispersion around their true values, an expected feature given that the sample period is less than two decades. Similarly, the jump parameters in $\theta_{\Pi}$ have much lower precision than the corresponding ones in $\theta_M$ because only a few jumps occur on average per year, whereas the variance swap rates contain substantial information about jump parameters in $\theta_M$. For the same reason, the estimates of the pricing error variances are also very precise.

4. Empirical results

4.1. Data

We estimate all six models using daily S&P 500 index returns and variance swap rates with six different maturities (2, 3, 6, 9, 12, and 24 months) over the period from January 4, 1996 to January 11, 2013. The number of daily observations is 4,276, excluding weekends and holidays. Due to restrictions from our data source, the sample is constructed as follows: it contains data on variance swap mid-quotes on 5 maturities (2, 3, 6, 12 and 24 months) from an anonymous U.S. bank over the period January 4, 1996 to
March 30, 2007, whereas the second dataset, which belongs to the same source, covers the period starting from January 2, 2001 to January 11, 2013 with 4 maturities (3, 6, 9, and 12 months). Overall, we have an unbalanced panel of variance swaps over the past 17 years.

Fig. 2 presents the variance swap rates for different maturities along with S&P 500 index returns over the whole sampling period. During the first half of the sample, they are characterized by a significantly higher market volatility which is due in part to the Asian, Russian and LTCM crises. After the quiet period between March 30, 2007, whereasthe second dataset, which belongsto the second dataset, which belongs to the same source, covers the period starting from January 2, 2001 to January 11, 2013 with 4 maturities (3, 6, 9, and 12 months). Overall, we have an unbalanced panel of variance swaps over the past 17 years.13

The bottom panel of Fig. 2 highlights the changes in the slope of the variance term structure. For most of the sample, the variance term structure is upward sloping, whereas in the middle and aftermath of crises, the term structure switches to a downward sloping shape, suggesting that volatility is expected to decrease towards its long-term mean level. The fact that the term structure is not in perfect tandem with the variance level suggests the necessity of incorporating at least one additional factor that captures the slope of the term structure of variance. In the next section, we document a few empirical facts that surface from our analysis.

4.2. Model performance

4.2.1. Choice of models

We perform principal component analysis for the balanced panel with 1558 observations. The first three eigenvalues account for 97.80%, 99.69%, and 99.91% of the total variations. The corresponding eigenvectors suggest that the first principal component is related to level shifts in the variance curve while the second one captures changes in the slope of the curve. Nevertheless, the convexity effect seems negligible for variance swap data, as the contribution of the third principal component is tiny.

While our principal component analysis suggests that one factor can explain a good deal of variation in variance swap prices, Aït-Sahalia et al. (2015) find strong support for two-factor models through a more formal comparison based on the likelihood ratio criterion for non-nested models. For this reason, in what follows we focus our analysis on alternative models that have two volatility factors.14 In light of the evidence of negative jumps highlighted in Fig. 1, we estimate models $A_0(2)$ and $A_1(2)$, each of which allows for negative jumps through at least one Ornstein–Uhlenbeck factor, as well as model $A_0(2)$, despite its inadequacy of capturing negative volatility jumps. For additional comparison, we also fit $A_{01}^\uparrow(2)$ and $A_{01}^\downarrow(2)$, which are special cases of non-affine models without downward jumps, as well as $A_2(2)$, the affine model prevalent in the literature. This comparison will shed light on the importance of downward volatility jumps and the advantage of non-affine models.

4.2.2. Estimation results

In Tables 2–5, we report the posterior means and standard deviations of the parameter vectors $\Theta_M$, $\Theta_H$, $\Theta_P$, and $\Theta_\Theta$ for all six models. Parameters are defined in annual terms following the convention in the empirical option pricing literature.

We first discuss the estimates of $\Theta_M$ and $\Theta_H$. As can be seen from the $k_{11}$’s estimates, the first factor $X_t$ mean-reverts much faster than the second factor $X_t$. $\kappa_{12}$ is closer to the values found in the option pricing literature under the pricing measure. Also, the mean reversion parameter of $X_2$ under both measures is very low, around 0.2 for both $A_{01}^\uparrow(2)$ and $A_{11}^\uparrow(2)$, implying that shocks to $X_2$ have a half life of several years. Moreover, for both $A_0(2)$ and $A_1(2)$ models, positive jump sizes are larger under $Q$ than under $P$, while negative jump sizes are smaller in magnitude under $Q$. Their differences seem statistically significant, which indicates that both types of jumps are priced. The lower panel of Table 2 contains the corresponding summary statistics of the posterior distribution of the pricing equation parameters in $\Theta_P$. Not surprisingly, these parameters are estimated with high precision given that they are identified from prices. Moreover, the percentage of volatility explained by the exponential component dominates, accounting for on average over 90% across all models. This provides strong evidence in favor of log-type volatility models against the affine volatility models or the quadratic ones advocated by Filipovic et al. (2016).

The persistence of volatility and its zero-lower bound together impose a substantial barrier for fitting square-root models. As a

13 While we do not have data on 1-month variance swaps, which may help identify jumps from stochastic volatility and which are informative about risk premia as shown by Andersen et al. (2017), we use the squared-VIX in out-of-sample studies and find very small pricing errors.

14 Andersen et al. (2015b) recently propose a three-factor parametric model for S&P 500 options. It would be interesting to compare the performance of two-factor non-affine models with that of three-factor affine models and assess whether adding a third factor would improve the pricing performance of affine models. Clearly, this question should be addressed using S&P 500 options or VIX options. We leave such an exercise for future work.
result, it is not surprising to find that the estimated parameters violate the boundary nonattainment conditions given explicitly in Section 2.3 and Supplemental Appendix D for both \( h_2(2) \) and \( A_2(2) \) models. In contrast, the parameter estimates from the other four models satisfy all the required conditions.

Figure A.1 provides the time series of the estimated factors for all six models. For \( h_2(2) \), we plot the estimated factors in log scales for comparison. Interestingly, the extracted factors (or their logarithms) share similar patterns across all models. Although the levels of these factors are not the same due to the sign restrictions, the similarity suggests that the extracted patterns are very robust. The factors from the \( h_2(2) \) model appear noisier over 2004–2007 when volatility is persistently small. This is because the variance swap rates are not sensitive to the magnitude of volatility factors, when they are at a extremely low level as required by the fitting and specification of this model.

Overall, the above analysis suggests that models on which at least one Ornstein–Uhlenbeck factor are more desirable than those with two square-root-factors.

4.2.3. Assessment of the \( Q \)-measure dynamics

We then analyze the properties of pricing error variances \( \theta_q \), from which we can intuitively learn about the performance of different models. Ideally, better models tend to produce smaller pricing errors, given similar amounts of unknown parameters. It turns out that we find very similar performances across the non-affine models. As shown in Tables 4 and 5, the estimated standard deviations of pricing errors are around 0.37, 0.07, 0.21, 0.25, 0.06, and 0.23 for the 6 maturities, respectively, with \( A_0(2) \) and \( h_1(2) \) being slightly better. For the affine model, the corresponding numbers are larger across the board: 0.38, 0.08, 0.23, 0.26, 0.06, and 0.27. In short, non-affine models achieve a better fitting parsimoniously.

We then compare the out-of-sample performance of model-fitting using the VIX. The out-of-sample study here is cross-sectional, instead of based on time-series forecasting, as is common for models with latent factors, see Piazzesi (2010). The VIX is constructed by the CBOE using option portfolios, which often coincides with how variance swap contract writers hedge their risk exposure. As a result, it is expected that the time-series of the squared VIX (scaled by 100) and the model-predicted 1-month swap rates present very similar patterns.\(^{15}\) The results are shown

\[ \text{Table 2} \]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>( A_0(2) )</th>
<th>( h_1(2) )</th>
<th>( A_2(2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda^2 )</td>
<td>Mean</td>
<td>Stddev</td>
<td>HPD 95%</td>
</tr>
<tr>
<td>( \lambda^2 )</td>
<td>7.519</td>
<td>0.117</td>
<td>[7.369, 7.717]</td>
</tr>
<tr>
<td>( \lambda^2 )</td>
<td>-4.322</td>
<td>0.048</td>
<td>[-4.390, -4.227]</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>2.001</td>
<td>0.063</td>
<td>[1.918, 2.116]</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>-0.225</td>
<td>0.005</td>
<td>[-0.233, -0.213]</td>
</tr>
</tbody>
</table>

Note: This table presents the posterior estimates for \( \theta_q \), the parameters determining the dynamics of the latent factors under the risk-neutral measure, and \( \theta_q \), the parameters defining \( f \) across all models. We report the mean, the standard deviation, and the 95% highest posterior density intervals for the \( A_0(2), h_1(2) \) and \( A_2(2) \). We use daily data on variance swaps from January 4, 1996 to January 1, 2011. The number of daily observations is 4276, excluding weekends and holidays.


15 Strictly speaking, the difference between the squared VIX and variance swap rates is related to the higher order moments of price jumps, as Aït-Sahalia et al. (2014) point out.
Fig. 2. The S&P 500 index and variance swap rates. Note: The top panel plots the time series of the S&P 500 index and its returns from January 4, 1996 to January 11, 2013. The second panel shows the variance swap rates with 6 different maturities. The maximum number of daily observations is 4276, excluding weekends and holidays. Since we have an unbalanced panel of variance swaps, different maturities may have different number of observations, which are reported in the legend. The bottom panel plots the slopes of corresponding variance swap rates, i.e.,

\[ P(t, \frac{1}{2}) - P(t, 1) - P(t, \frac{1}{4}) \] and

\[ P(t, 1) - P(t, \frac{1}{4}) - P(t, \frac{1}{4}) \]

respectively, where \( P(t, \tau) \) denotes the variance swap rate at time \( t \) of a contract with time to maturity \( \tau \). Positive values reflect an upward sloping term structure while the opposite slope is implied by negative values.

in Figure A.2. The out-of-sample performance compared to the VIX is almost identical across these models, with correlations as high as 0.89 for all.

Finally, we conduct a Diebold–Mariano style test, see Diebold and Mariano (1995), to compare the performance of competing models. For simplicity, we assume all estimation errors are covariance stationary, so that we can use their Gaussian large-sample critical values. Our time series of pricing errors do seem to support this claim, though as Diebold (2015) suggests, this assumption is not necessarily required in many settings. We use the squared pricing errors at each time point as our loss function, for each of the six time-to-maturities, respectively. We adopt Newey–West estimator to standardize the difference of the loss functions. Table 6 reports the 95% confidence bands based on 1000 draws from the posterior distribution of parameter and latent factors, for pairwise model-comparison test statistics with null hypothesis being no differences between the pair of models. Test statistics taking positive (resp. negative) values, i.e., confidence bands staying on the right (resp. left) side of 0, provide evidence against (resp. supporting) the benchmark model. Our results provide strong evidence against \( A_2(2) \) and its affine version \( \bar{A}_2(2) \), yet find a close tie between \( A_0(2) \) and \( A_1(2) \).

4.2.4. Assessment of the P-measure dynamics

Having witnessed quite similar results across the Q-measure performance of these non-affine models, in particular the \( A_0(2) \) and \( A_1(2) \) models, we then move on to their P-measure performance by investigating the time series of the estimated spot variance \( \sigma_t^2 \), which can be decomposed into jumps and Brownian shocks. We decompose changes of estimated spot variances for all six models in Fig. 3, respectively, which sheds light on some new evidence on model selection among two-factor volatility models.

The changes of the spot variance are very similar to changes of the squared VIX in Fig. 1 across all models (hence we omit these figures), but their decompositions are strikingly different. We highlight on the figure three downward volatility jumps associated with the three news events mentioned at the beginning of the introduction. Obviously, except for the \( A_0(2) \) and \( A_1(2) \) models, the rest cannot capture any of these downward volatility jumps, so that they are misidentified as large Brownian shocks. Although the \( A_2(2) \) model is able to capture negative jumps in one of its factors, the Ornstein–Uhlenbeck one, this factor turns out to be slow mean-reverting and highly persistent, which cannot accommodate those jumps that perhaps only affect short-term volatility levels. As a result, several significantly downward volatility changes are attributed to Brownian shocks, as the square-root factor does not permit negative jumps. Although the \( A_1(2) \) model is able to capture negative jumps in one of its factors, the Ornstein–Uhlenbeck one, this factor turns out to be slow mean-reverting and highly persistent, which cannot accommodate those jumps that perhaps only affect short-term volatility levels. As a result, several significantly downward volatility changes are attributed to Brownian shocks, as the square-root factor does not permit negative jumps. As previously mentioned, for the \( A_2(2) \) model, the Feller constraint is binding for both the P- and Q-measure dynamics. In contrast, the \( A_0(2) \) model can accommodate jumps in both the short-term and long-term factors, so that those short-term jumps missed by \( A_1(2) \) are captured, and that the two components are well-separated. Overall, the \( A_0(2) \) model is clearly more desirable from the evidence in Fig. 3.

We hence employ the \( A_0(2) \) model in the following empirical analysis. While we can further improve the \( A_0(2) \) model by augmenting it with a time-varying intensity factor, we choose not to do so for parsimony and in order to avoid overfitting.

4.3. Economic interpretation of volatility components

4.3.1. Volatility factors

We now provide an interpretation for the latent volatility factors, before addressing how jumps are related to them. To do so, we conduct a regression analysis trying to link the identified latent factors with factors of economic fundamentals.\(^{16}\) We select

\(^{16}\) It is worth mentioning that labeling latent factors using time series regressions is perhaps not compelling for all sorts of reasons, despite it is common practice in
two credit variables, including the daily TED spread, calculated as the difference between the three-month LIBOR and the three-month T-Bill interest rate and the default spread (DEF), calculated as the difference between the monthly Moody’s AAA and BAA corporate bond yield. We also obtain two monthly macroeconomic factors from the Federal Reserve’s website: the Chicago Fed National Activity Index (CFI), constructed from 85 monthly indicators of economic activity; and industrial production growth (IPG), as suggested by Pástor and Veronesi (2013) and Adrian and Rosenberg (2008). We also include the daily term spread (TERM), i.e. the difference between the yields on the 10-year and 3-month Treasury securities. We also add one monthly liquidity factor (LIQ), the innovation of the aggregate liquidity from Pástor and Stambaugh (2003). To identify potential policy risk that may be related to volatility jumps, we add the policy news index (POL) constructed by Baker et al. (2013). Finally, we construct the market skewness factor as it is shown to be important for the short-term component by Adrian and Rosenberg (2008).

We consider one-by-one simple regressions of the posterior mean of each factor $X_t$ of $A_0(2)$, sampled at the end of each month from 1996 to 2012, on the innovation of each covariate given above, as well as the lagged value of the posterior mean of $X_t$ from the past month:

$$X_{t,i} = \beta_0 + \beta_i Z_{t,i} + \beta_2 X_{t-1,i} + \epsilon_{t,i},$$

with $Z_{t,i}$ being the innovation of the $j$th covariate. For POL and TERM, we use ARIMA(1,1,0) innovations, as the Dickey Fuller tests fail to reject the unit-roots in our sample period. For IPG, we use the AR(3) innovation, following Adrian and Rosenberg (2008). For the rest of the covariates, we use AR(1) innovations. The results are qualitatively identical when using other regression specifications.

We also consider a multiple time-series regression for all the innovations of the covariates plus the lagged posterior mean of $X_t$:

$$X_{t,i} = \beta_0 + \beta_1 \text{DEF}_t + \beta_2 \text{TED}_t + \beta_3 \text{TERM}_t + \beta_4 \text{LIQ}_t + \beta_5 \text{POL}_t + \beta_6 \text{SKEW}_t + \beta_7 \text{EXM}_t + \beta_8 \text{IPG}_t + \beta_9 \text{CFI}_t + \beta_{10} X_{t-1,i} + \epsilon_{t,i}. \tag{11}$$

Tables 7 and 8 provide regression results for $X_1$ and $X_2$, respectively. Table 7 suggests that the time variation of short-term volatility factor $X_1$ is associated with credit risk, liquidity risk, and policy news, in addition to the excess returns. The signs of each coefficient agree with the intuition that short-term volatility rises if risk or uncertainty increases. When stacking these covariates into the multiple regression, policy news, excess market returns, and lagged values of $X_1$ subsume the rest of the covariates. As for the second volatility factor $X_2$, default risk, term premium, and excess market return become significant with all covariates included. The AR(1) coefficient reported in Table 8 confirms that $X_2$ is much more persistent than $X_1$. It is worth mentioning that our business cycle variables are not significant, for potentially two reasons. First, the sample period is as short as 17 years, which does not accommodate
The number of daily observations is 4276, excluding weekends and holidays.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\hat{\lambda}_d(2)$</th>
<th>$\hat{\lambda}_j(2)$</th>
<th>$\hat{\lambda}_y(2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.249</td>
<td>1.004</td>
<td>0.962</td>
</tr>
<tr>
<td>Stddev</td>
<td>[−1.278, −0.295]</td>
<td>[6.601, 10.530]</td>
<td>[7.000, 10.703]</td>
</tr>
</tbody>
</table>

Table 4

Posterior estimates of $\theta_2$ and $\theta_y$ for $\hat{\lambda}_d(2)$, $\hat{\lambda}_j(2)$, and $\hat{\lambda}_y(2)$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\hat{\theta}_2$</th>
<th>$\hat{\theta}_y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.232</td>
<td>0.129</td>
</tr>
<tr>
<td>Stddev</td>
<td>0.030</td>
<td>0.018</td>
</tr>
<tr>
<td>HPD 95%</td>
<td>[0.179, 0.297]</td>
<td>[0.100, 0.168]</td>
</tr>
</tbody>
</table>

Note: This table presents the posterior estimates for the parameters that characterize the $T$-dynamics, $\theta_2$, and the pricing error variances $\theta_y$. We report the mean, the standard deviation, and the 95% highest posterior density intervals for the $\hat{\theta}_2$ and $\hat{\theta}_y$. We use daily data on variance swaps from January 4, 1996 to January 11, 2013. The number of daily observations is 4276, excluding weekends and holidays.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\hat{\theta}_2$</th>
<th>$\hat{\theta}_y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.143</td>
<td>0.044</td>
</tr>
<tr>
<td>Stddev</td>
<td>0.004</td>
<td>0.001</td>
</tr>
<tr>
<td>HPD 95%</td>
<td>[0.135, 0.151]</td>
<td>[0.104, 0.105]</td>
</tr>
</tbody>
</table>

Table 5

Posterior estimates of $\theta_2$ and $\theta_y$ for $\hat{\lambda}_d^*(2)$, $\hat{\lambda}_j^*(2)$, and $\hat{\lambda}_y^*(2)$.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\hat{\lambda}_d^*(2)$</th>
<th>$\hat{\lambda}_j^*(2)$</th>
<th>$\hat{\lambda}_y^*(2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.228</td>
<td>1.052</td>
<td>0.024</td>
</tr>
<tr>
<td>Stddev</td>
<td>[−1.310, −0.237]</td>
<td>[6.393, 10.539]</td>
<td>[0.072, 0.165]</td>
</tr>
<tr>
<td>HPD 95%</td>
<td>[−1.691, −0.877]</td>
<td>[13.851, 7.471]</td>
<td>[1.174, 0.610]</td>
</tr>
</tbody>
</table>

Note: This table presents the posterior estimates for the parameters that characterize the $T$-dynamics, $\theta_2$, and the pricing error variances $\theta_y$. We report the mean, the standard deviation, and the 95% highest posterior density intervals for the $\hat{\lambda}_d^*(2)$, $\hat{\lambda}_j^*(2)$, and $\hat{\lambda}_y^*(2)$. We use daily data on variance swaps from January 4, 1996 to January 11, 2013. The number of daily observations is 4276, excluding weekends and holidays.
many business cycles. Secondly, the longest maturity of our variance swaps is 2 years, so that the “long” term factor extracted here may be regarded as the “medium” term in macroeconomics, so that business cycle variables are less important. The results are very similar for the other non-affine models, an expected feature in light of Figure A.1 of the supplemental appendix.

4.3.2. Volatility jumps

Regarding volatility jumps, we find that downward volatility jumps are as common as positive ones, and that they are often associated with a resolution of policy uncertainty. Apart from the three news headlines mentioned in the introduction, we highlight 29 additional days in Table 9, which are clearly related to some policy news, out of the 40 days with largest downward volatility jumps. From this table, we find that the majority of large downward volatility jumps are associated with changes of current monetary policy or clear indications about future monetary policy, despite few jumps being relevant to fiscal policy, all of which may help comfort investors.

<table>
<thead>
<tr>
<th>Time-to-maturity</th>
<th>2 months</th>
<th>3 months</th>
<th>6 months</th>
<th>9 months</th>
<th>1 year</th>
<th>2 years</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda_0(2)$</td>
<td>[0.45, 2.31]</td>
<td>[3.49, 10.19]</td>
<td>[2.33, 2.90]</td>
<td>[3.75, 5.92]</td>
<td>[−0.69, 6.38]</td>
<td>[2.17, 2.93]</td>
</tr>
<tr>
<td>$\Lambda_1^+(2)$</td>
<td>[0.46, 2.15]</td>
<td>[2.17, 9.40]</td>
<td>[2.43, 2.96]</td>
<td>[4.23, 6.02]</td>
<td>[−6.49, −1.03]</td>
<td>[2.30, 3.08]</td>
</tr>
<tr>
<td>$\Lambda_2(2)$</td>
<td>[−0.28, 1.74]</td>
<td>[4.64, 11.24]</td>
<td>[2.46, 3.01]</td>
<td>[3.59, 5.92]</td>
<td>[3.13, 9.87]</td>
<td>[2.40, 3.05]</td>
</tr>
<tr>
<td>$\Lambda_1^-(2)$</td>
<td>[−0.40, 1.49]</td>
<td>[4.41, 10.79]</td>
<td>[2.52, 3.06]</td>
<td>[3.67, 6.00]</td>
<td>[−2.11, 5.28]</td>
<td>[2.40, 3.15]</td>
</tr>
<tr>
<td>$\Lambda_2(2)$</td>
<td>[0.16, 2.37]</td>
<td>[6.65, 46.67]</td>
<td>[2.61, 3.98]</td>
<td>[9.49, 6.5]</td>
<td>[−9.78, 3.31]</td>
<td>[0.58, 2.71]</td>
</tr>
</tbody>
</table>

Note: This table presents the 95% confidence bands based on 1000 posterior draws for Diebold and Mariano (1995) style test statistics in a pair-wise comparison of the pricing errors among alternative models. For each panel, test statistics taking positive (resp. negative) values, i.e., confidence bands staying on the right (resp. left) side of 0, provides evidence against (resp. supporting) the benchmark model.

<table>
<thead>
<tr>
<th>Variable</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
<th>(8)</th>
<th>(9)</th>
<th>(10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DEF</td>
<td>0.278***</td>
<td>(0.107)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.091</td>
</tr>
<tr>
<td>TED</td>
<td>0.184*</td>
<td>(0.099)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.078</td>
</tr>
<tr>
<td>TERM</td>
<td>−0.003</td>
<td>(0.076)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>−0.008</td>
</tr>
<tr>
<td>LIQ</td>
<td>−0.544**</td>
<td>(0.217)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.126</td>
</tr>
<tr>
<td>POL</td>
<td></td>
<td></td>
<td>0.004***</td>
<td>(0.001)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.002**</td>
<td>(0.001)</td>
</tr>
<tr>
<td>SKEW</td>
<td></td>
<td>0.018</td>
<td>(0.032)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.033</td>
<td></td>
</tr>
<tr>
<td>ExM</td>
<td></td>
<td></td>
<td>−0.027***</td>
<td>(0.003)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>−0.025***</td>
<td>(0.004)</td>
</tr>
<tr>
<td>IGF</td>
<td></td>
<td></td>
<td>−0.011</td>
<td>(0.037)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>−0.032</td>
<td>(0.033)</td>
</tr>
<tr>
<td>CFI</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.012</td>
<td>(0.029)</td>
<td></td>
<td></td>
<td>0.061</td>
<td></td>
</tr>
<tr>
<td>AR</td>
<td>0.769***</td>
<td>(0.037)</td>
<td>0.775***</td>
<td>(0.036)</td>
<td>0.780***</td>
<td>(0.038)</td>
<td>0.765***</td>
<td>(0.035)</td>
<td>0.791***</td>
<td>(0.033)</td>
</tr>
<tr>
<td>adj. $R^2$</td>
<td>0.608</td>
<td>0.615</td>
<td>0.6</td>
<td>0.611</td>
<td>0.64</td>
<td>0.601</td>
<td>0.716</td>
<td>0.601</td>
<td>0.601</td>
<td>0.73646</td>
</tr>
</tbody>
</table>

Note: In this table we report results from regression analysis to relate our volatility factors to several variables on economic fundamentals at a monthly frequency. Each column reports the results of estimating a linear regression of the posterior mean of the volatility factor $X_1$ on its lagged value (AR) and the innovation of the corresponding variable for each row. The last column corresponds to the multiple regression that includes all the explanatory variables we consider: TED spread, default spread (DEF), Chicago Fed National Activity Index (CFI), industrial production growth (IPG), term spread (TERM), monthly liquidity factor (LIQ), policy news index (POL), market skewness (SKEW), and excess market returns (ExM), see Section 4.1 for more information on their definitions. The coefficients corresponding to the constants are omitted from the regressions.

Table 6
Comparison of pricing errors.

Table 7
Regression results on factor $X_1$. 

To understand how volatility jumps originate, we construct measures of news surprises based on surveys of economists’ expectations on 18 economic indicators from Bloomberg. The detailed information about the categories, the announcement time, and the frequency of these news events are given in Table A.4 of the supplemental appendix. We proxy news surprise as the scaled differences between the actual news release and the median of survey expectations:

\[
\text{News Surprise} = \frac{\text{Announced Quantity} - \text{Median of Expectations}}{\text{Maximum of Expectations} - \text{Minimum of Expectations}}. \tag{12}
\]

The news surprises of economic indicators are treated as the control variables since they are expected to produce jumps in S&P 500 returns and other markets, e.g. Beechey and Wright (2009) and Faust and Wright (2009). To proxy the resolution of policy uncertainty, we use the schedules of FOMC and ECB meetings, as well as the speech schedules of Federal Reserve’s Chairman. Although the schedules are usually pre-announced and the target interest rates do not change often, the minutes, statements or press conferences after the FOMC meetings are informative about monetary policy decisions, and this information could be unpredictable.

We regress the magnitudes of positive and negative volatility jumps onto the magnitude of macroeconomic news surprises for each volatility factor, respectively\(^\text{17}\):

\[
|\text{Positive/Negative jump size of } X_{j,t} | = \beta_{j,0}^{+/-} + \sum_{i=1}^{21} \beta_{j,i}^{+/-} s_{i,t} + \epsilon_{j,t}^{+/-}, \quad \text{for } j = 1, 2 \tag{13}
\]

where coefficients with +/- correspond to regressions with positive and negative jumps, respectively, and \(s_{i,t}\) is the ith news surprise at time \(t\). If there is no such news event on day \(t\), then \(s_{i,t}\) is set to 0. All news surprises are rescaled so that they all have time-series standard deviation equal to 1.

In addition, we also run two similar regressions for jumps in \(\sigma_t^2\):

\[
|\text{Positive/Negative jump size of } \sigma_t^2 | = \beta_{0}^{+/-} + \sum_{i=1}^{21} \beta_{i}^{+/-} s_{i,t} + \epsilon_{t}^{+/-}. \tag{14}
\]

The results are provided in Table 10. We find that FOMC meetings and Federal Reserve Chairmen’s speeches are related to negative volatility jumps of \(X_t\) and \(\sigma_t^2\), whereas other volatility jumps are

\(^{17}\) We use the posterior medians of the identified jumps so as to obtain a more sparse time series of jumps.
related to surprising news about employment, consumer spending, and national output. This conforms with our conjecture and earlier event studies suggesting that negative volatility jumps are associated with the resolution of policy uncertainty. In addition, such jumps mostly affect the short-term volatility level, suggesting that not all policy measures have a significant impact on long-term uncertainty.

To further analyze the short-term versus long-term impact of policy news, we conduct event studies by investigating the identified jumps in X1 and X2 separately, for the three events mentioned in the introduction. It turns out that not all of the three policy news we highlighted have a strong long-term impact on volatility, despite their significant influences on the short-term volatility level with magnitudes (posterior medians) as high as 0.37, 0.24, and 0.29, respectively. Regarding Europe’s Debt Crisis, the unprecedented emergency loan plan unveiled on May 10, 2010 hit the long-term volatility level by −0.09, although a larger downward jump of magnitude 0.29 in X2 came two days later, as investors digested the details of the $1 trillion European aid package. Another long-term volatility jump (−0.24) came more than one year later, after European Union leaders agreed to expand Europe’s bailout fund and take major losses on Greek bonds at the end of marathon talks on October 27, 2011. Also, the Federal Reserve’s FOMC statement on August 9, 2011 decreases the long-term uncertainty level by −0.10, potentially because of the additional “forward guidance” information on how long the Committee expects to keep the target for the federal funds rate exceptionally low. In contrast, the news about the fiscal cliff do not show a significant impact, as investors remained cautious about the deal. Indeed, Congress failed to reach an agreement on spending cuts and the sequestration was delayed until March 2013 as part of the American Taxpayer Relief Act of 2012, passed by Congress on January 1, 2013.

4.4. Variance risk premia

Now we investigate the pricing implications of downward volatility jumps. Comparing the estimates in Table 2, positive volatility jumps have larger magnitudes under the Q-measure than under the P-measure, whereas negative jumps have smaller magnitudes under the Q-measure, which suggests that both positive and negative jumps are priced. This implies that market participants are perhaps pessimistic about the impact of the anticipated bailout, the size and scope of which are often uncertain. The exact risk premia due to negative jumps are difficult to disentangle from the total risk compensation, because their inclusion affects realizations of factors, their expectations, and hence nearly everything. To gauge their economic impact, we compare the total variance risk premia implied from $A_0(2)$ with the estimates from $A_0^+(2)$ model, which does not include downward volatility jumps. The difference in the amount of variance risk premia can be interpreted as the bias due to omission of downward volatility jumps.

We define variance risk premia in the same way as was introduced in Bollerslev et al. (2009), Carr and Wu (2009), and Todorov (2010):

\[
VRP(t, \tau) = \frac{1}{T} \left[ \mathbb{E}_T^Q \{ Y, Y_{t+1}^T \} - \mathbb{E}_T^P \{ Y, Y_{t+1}^T \} \right].
\]

We plot in Fig. 4 the time series of the term structure of variance risk premia implied from the $A_0(2)$ and $A_0^+(2)$ models, respectively. The plots suggest that variance risk premia are mostly negative, with confidence bands not including zero, and countercyclical, (i.e., they become even more negative in bad times). For example, the lower troughs in the figure are associated with the 1997–1998 Asian crisis, the dot-com bubble, the recent financial meltdown, and the European and U.S. debt crises, suggesting that investors require more compensation for bearing variance risk during difficult times.

However, what we find more striking is that at the inception of the aforementioned crises, the estimates of variance risk premia become positive or at least insignificantly different from zero (the shaded areas cross zero) for short periods of time based on $A_0(2)$, whereas for $A_0^+(2)$ the estimates are almost always negative. The finding based on $A_0(2)$ agrees with most model-free estimates, see Table A.1 of the supplemental appendix.\(^{18}\) Our result also suggests that $A_0(2)$ with downward volatility jumps allows for more flexible specification of the pricing kernel. It does not rule this pattern out

\(^{18}\) Note that model-free estimates still rely on affine volatility forecasting models to estimate the conditional variance under the objective measure.
Table 9
Policy news potentially associated with estimated volatility jumps.

<table>
<thead>
<tr>
<th>Date</th>
<th>Date Var</th>
<th>News</th>
</tr>
</thead>
<tbody>
<tr>
<td>08/18/98</td>
<td>−0.365</td>
<td>FOMC’s Decision to Leave Interest Rates Unchanged</td>
</tr>
<tr>
<td>09/01/98</td>
<td>−0.664</td>
<td>Fed Adds Money to the Banking System with Repo</td>
</tr>
<tr>
<td>09/08/98</td>
<td>−0.455</td>
<td>Fed Chairman A. Greenspan’s Statement that a Rate Cut might be Forthcoming</td>
</tr>
<tr>
<td>09/14/98</td>
<td>−0.185</td>
<td>President Clinton Advocated a Coordinated Global Policy for Economic Growth in NYC</td>
</tr>
<tr>
<td>09/23/98</td>
<td>−0.280</td>
<td>Fed Chairman A. Greenspan Testimony Before the Committee on the Budget, U.S. Senate</td>
</tr>
<tr>
<td>08/11/99</td>
<td>−0.276</td>
<td>Fed Beige Book Release Shows that US Economy Remains Strong</td>
</tr>
<tr>
<td>04/17/00</td>
<td>−0.296</td>
<td>Treasury Secretary L. Summers Statement that Fundamentals of Economy are in Place</td>
</tr>
<tr>
<td>01/03/01</td>
<td>−0.179</td>
<td>Fed’s Announcement of a Surprise, Inter-Meeting Rate Cut</td>
</tr>
<tr>
<td>05/17/05</td>
<td>−0.303</td>
<td>John Snow Call on China to Take An Intermediate Step in Revaluing its Currency</td>
</tr>
<tr>
<td>05/19/05</td>
<td>−0.276</td>
<td>Fed Chairman A. Greenspan Steps up Criticism of Fannie Mae and Freddie Mac</td>
</tr>
<tr>
<td>06/15/06</td>
<td>−0.625</td>
<td>Fed’s Announcement Generated Market Rebound the Previous Day</td>
</tr>
<tr>
<td>06/29/06</td>
<td>−0.325</td>
<td>FOMC Statement to Raise Its Target for the Federal Funds Rate by 25 Basis Points</td>
</tr>
<tr>
<td>07/19/06</td>
<td>−0.272</td>
<td>Fed Chairman B. Bernanke Warned that the Fed Must Guard Against Rising Prices Taking Hold</td>
</tr>
<tr>
<td>02/28/07</td>
<td>−0.396</td>
<td>Fed Chairman B. Bernanke Told a House Panel that Markets Seemed Working Well</td>
</tr>
<tr>
<td>03/06/07</td>
<td>−0.217</td>
<td>Henry Paulson in Tokyo Said the Global Economy was As Strong As He’s Ever Seen</td>
</tr>
<tr>
<td>03/21/07</td>
<td>−0.244</td>
<td>Fed Policy Makers Concluded their Two-Day Policy Meeting by</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Keeping the Fed Fund Rate</td>
</tr>
<tr>
<td>06/27/07</td>
<td>−0.271</td>
<td>FOMC Announcement Generated Market Rebound the Previous Day</td>
</tr>
<tr>
<td>08/21/07</td>
<td>−0.188</td>
<td>Senator Dodd said the Fed to Deal with the Turmoil after Meeting with Paulson and Bernanke</td>
</tr>
<tr>
<td>09/18/07</td>
<td>−0.353</td>
<td>FOMC Decided to Lower its Target for the Federal Funds Rate by 50 Basis Points</td>
</tr>
<tr>
<td>03/18/08</td>
<td>−0.216</td>
<td>Fed Cut the Federal Funds Rate by Three-Quarters of a Percentage Point</td>
</tr>
<tr>
<td>10/14/08</td>
<td>−0.304</td>
<td>President Bush and Henry Paulson Separately Announced Revisions to the TARP Program</td>
</tr>
<tr>
<td>10/20/08</td>
<td>−0.413</td>
<td>Fed Chairman B. Bernanke Testimony on the Budget, U.S. House of Representatives</td>
</tr>
<tr>
<td>10/28/08</td>
<td>−0.230</td>
<td>Fed to Cut the Rate Following the Two-Day FOMC Meeting is Expected by the Market</td>
</tr>
<tr>
<td>11/13/08</td>
<td>−0.240</td>
<td>President Bush’s Speech on Financial Crisis</td>
</tr>
<tr>
<td>12/19/08</td>
<td>−0.244</td>
<td>President Bush Declared that TARP Funds to be Spent on Programs Paulson Deemed Necessary</td>
</tr>
<tr>
<td>01/21/09</td>
<td>−0.206</td>
<td>T. Geithner Testified about Nomination as Treasury Secretary before the Senate Finance Committee</td>
</tr>
<tr>
<td>02/24/09</td>
<td>−0.261</td>
<td>President Obama’s First Speech as the President to Joint Session of U.S. Congress</td>
</tr>
<tr>
<td>05/10/10</td>
<td>−0.601</td>
<td>European Policy Makers Unveiled An Unprecedented Emergency Loan Plan</td>
</tr>
<tr>
<td>03/21/11</td>
<td>−0.277</td>
<td>Japanese Nuclear Reactors Cooled Down and Situations in Libya Tamed by Unilateral Forces</td>
</tr>
<tr>
<td>08/09/11</td>
<td>−0.370</td>
<td>FOMC Statement Explicitly Stating A Duration for An Exceptionally Low Target Rate</td>
</tr>
<tr>
<td>10/27/11</td>
<td>−0.205</td>
<td>European Union Leaders Made a Bond Deal to Fix the Greek Debt Crisis</td>
</tr>
<tr>
<td>01/02/13</td>
<td>−0.427</td>
<td>President Obama and Senator McConnell’s Encouraging Comments on the “Fiscal Cliff” Issue</td>
</tr>
</tbody>
</table>

Note: In this table, we report the 32 potential events in the last column that may lead to the 40 largest negative volatility jumps in sample. The first column is the date of the event, and the second column shows changes in estimated spot variance. The remaining dates on which policy related news could not be related to downward jumps are May 28, Oct 15, Oct 20, and Oct 30 of 1998, Jan 7 of 2000, June 1 of 2005, July 30 of 2007, and Nov 13 of 2007.

A close scrutiny of the dates on which our estimated variance risk premia are positive in Fig. 4 reveals that these dates are always associated with extreme market downturns. For instance, on Oct 27, 1997, the so-called mini-crash, the DJIA plunged 554 points or 7.2%, amid ongoing worries about the Asian economies. On Aug 31, 1998, the DJIA plunged another 512 points, or 6.4%, the second largest one-day loss in the index’s history by then. Oct 9, 2002 was the bottom of the NASDAQ following the dot-com collapse, accumulating a 42.88% loss from the beginning of 2002. On Sep 29, 2008, Congress rejected TARP, sending the DJIA down 778 points, its single-worst point drop ever.

The risk premia should have been more negative, rather than being statistical insignificant or even positive, if the variance risk were priced by the representative agent in a rational expectation framework. It is perhaps worth mentioning that even though our model is flexible, it is still difficult to ensure that we can identify the actual expectation of volatility under the $\pi$-measure at each point in time. For example, one might worry that around the beginning of the financial crisis, investors had a lot more information beyond what is embedded in the variance swap rates or S&P 500 returns, which are not used in our estimation. That being said, if one
Table 10
Regression results on volatility jumps.

<table>
<thead>
<tr>
<th>News/events</th>
<th>Jumps of $X_1$</th>
<th>Jumps of $X_2$</th>
<th>Jumps of $\sigma^2_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Positive (N)</td>
<td>Negative (N)</td>
<td>Positive (N)</td>
</tr>
<tr>
<td>Unemployment Rate</td>
<td>0.642 (0.407)</td>
<td>0.073 (0.300)</td>
<td>0.078 (0.553)</td>
</tr>
<tr>
<td>ADP Employment Change</td>
<td>-0.096 (0.324)</td>
<td>-0.253 (0.239)</td>
<td>-0.477 (0.440)</td>
</tr>
<tr>
<td>Initial Jobless Claims</td>
<td>-0.230 (0.228)</td>
<td>0.040 (0.168)</td>
<td>0.169 (0.310)</td>
</tr>
<tr>
<td>Personal Income</td>
<td>-0.185 (0.525)</td>
<td>-0.225 (0.388)</td>
<td>0.809 (0.713)</td>
</tr>
<tr>
<td>Personal Spending</td>
<td>0.042 (0.667)</td>
<td>-0.211 (0.492)</td>
<td>-1.354 (0.905)</td>
</tr>
<tr>
<td>Advance Retail Sales</td>
<td>0.022 (0.517)</td>
<td>-0.129 (0.381)</td>
<td>0.008 (0.701)</td>
</tr>
<tr>
<td>Consumer Confidence</td>
<td>-0.966 (0.615)</td>
<td>-0.313 (0.453)</td>
<td>-0.649 (0.834)</td>
</tr>
<tr>
<td>GDP</td>
<td>0.158 (0.105)</td>
<td>-0.021 (0.077)</td>
<td>-0.066 (0.142)</td>
</tr>
<tr>
<td>Durable Goods Orders</td>
<td>0.751* (0.407)</td>
<td>-0.196 (0.300)</td>
<td>-0.381 (0.552)</td>
</tr>
<tr>
<td>ISM Manufacturing</td>
<td>-0.102 (0.580)</td>
<td>0.986** (0.428)</td>
<td>1.688** (0.787)</td>
</tr>
<tr>
<td>Chicago PMI</td>
<td>1.785*** (0.436)</td>
<td>0.099 (0.322)</td>
<td>0.129 (0.592)</td>
</tr>
<tr>
<td>Empire State Manufacturing</td>
<td>-0.128 (0.430)</td>
<td>0.379 (0.317)</td>
<td>0.422 (0.583)</td>
</tr>
<tr>
<td>Business Inventories</td>
<td>0.132 (0.541)</td>
<td>-0.335 (0.399)</td>
<td>0.292 (0.734)</td>
</tr>
<tr>
<td>Production and Utilization</td>
<td>-0.073 (0.193)</td>
<td>0.255* (0.142)</td>
<td>0.251 (0.262)</td>
</tr>
<tr>
<td>New Residential Sales</td>
<td>0.701* (0.394)</td>
<td>0.194 (0.291)</td>
<td>0.004 (0.535)</td>
</tr>
<tr>
<td>FOMC Meetings</td>
<td>-0.316 (0.271)</td>
<td>0.476** (0.200)</td>
<td>0.383 (0.367)</td>
</tr>
<tr>
<td>Fed Chairman's Speeches</td>
<td>-0.135 (0.147)</td>
<td>0.263** (0.109)</td>
<td>0.151 (0.200)</td>
</tr>
<tr>
<td>ECB Meetings</td>
<td>0.108 (0.232)</td>
<td>-0.217 (0.171)</td>
<td>-0.056 (0.315)</td>
</tr>
<tr>
<td>CPI</td>
<td>-0.718 (0.849)</td>
<td>-0.825 (0.626)</td>
<td>-1.483 (1.152)</td>
</tr>
<tr>
<td>PPI</td>
<td>0.287 (0.624)</td>
<td>-0.229 (0.460)</td>
<td>-0.431 (0.847)</td>
</tr>
<tr>
<td>Employment Cost Index</td>
<td>-0.693 (1.528)</td>
<td>-0.166 (1.127)</td>
<td>1.770 (2.074)</td>
</tr>
</tbody>
</table>

Note: In this table, we report the regressions of the magnitudes of the jumps in each volatility factor as well as jumps in total volatility, onto the magnitudes of news shocks, defined in Section 4.1. All the shocks are standardized to have variance equal to 1. The regressands are taken from the posterior median of the identified jumps in $X_1$, $X_2$, and $\sigma^2_t$, respectively, based on the $A_0(2)$ model.

Fig. 4. Term structure of variance risk premia. Note: In this figure we plot the term structure of variance risk premia for $A_0(2)$ and $A_0^+(2)$, respectively. The green solid lines plot the risk premia for the 2-month contracts, the blue dash-dotted line for the 6-month contracts, and the red dashed line for the 1-year contracts. The shaded areas around the lines plot the 95% confidence intervals. We mark specific dates on which our estimates of variance risk premia are positive.
were to interpret these estimates in a structural framework, one may resort to a model in which investors have heterogeneous beliefs, so that the variance risk premia could be either positive or negative depending on the prevalent view of the market. In that regard, Bakshi et al. (2015) suggest a U-shape volatility pricing kernel by exploring the link between the monotonicity of the pricing kernel and returns on VIX option portfolios. They further build a stylized model with heterogeneity in beliefs to account for the U-shape. On the other hand, using traders’ position data from CFTC, Cheng (2015) finds evidence indicating that time-varying demand from heterogeneous investors affects premia embedded in VIX futures, and that the low demand from dealers and unlevered asset managers help explain the low premia during these periods.

5. Conclusion

Motivated by recent news headlines about the dramatic changes of the VIX following the announcements of policy makers, our systematic investigation examines the sudden declines of market volatility. We find downward volatility jumps to be as common as positive ones, and that the majority of them are associated with FOMC announcements and the speeches of Federal Reserve Chairmen, showing the impact of Central Bank intervention, whereas only a small portion of downward volatility jumps are responses to surprising news about employment, consumer spending, and national output. This conforms with earlier event studies suggesting that negative volatility jumps are highly correlated with the resolution of policy uncertainty. Moreover, we find that while such jumps affect the short-term volatility level, not all of them have a significant impact on long-term volatility.

Our results indicate that both positive and negative jumps are priced. While a model without downward volatility jumps may be able to capture part of the jump risk premia through other components in the model, the interpretation would be entirely different. Once downward volatility jumps are incorporated, total variance risk premia tend to become less negative or even positive for certain periods of crises, which leads to a puzzle many nonparametric studies also document.

In order to model downward volatility jumps, this paper introduces a new non-affine modeling framework which extends the classification and characterization of term structure models to allow for jumps. Our canonical models nest square-root factors, Ornstein–Uhlenbeck factors, pure-jump factors with state-dependent intensity, self-exciting jumps, Lévy jumps, etc. We find that the log-type volatility model, which has been favored by financial econometricians in the past, with at least one Ornstein–Uhlenbeck factor and double exponential jumps yields the best performance in fitting variance swap prices and the volatility dynamics. Such a model can also be used to investigate S&P 500 options, which we leave for future work.

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Appendix A. Supplementary data

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References


