Supplement to "Resolution of Policy Uncertainty and Sudden Declines in Volatility"

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Abstract

This appendix contains details about variance swap pricing, invariant transformations, extended canonical forms, volatility models, likelihood estimation, as well as additional tables and figures.

A Variance Swap Pricing

Proof of Proposition 1. Recall that since X is affine, the generalized conditional characteristic function (GCCF) of X_s is defined below, for any $s \geq t$ with t fixed:

$$\Psi(s,t,u,X_t) = \mathbb{E}_t^{\mathbb{Q}}\Big[e^{u^\top X_s}\Big],$$

where $u \in \mathbb{C}^{\mathrm{N}}$. There exists a closed-form formula for the GCCF function given by Duffie et al. (2000):

$$\log\Big(\Psi(s,t,u,X_t)\Big) = A(s-t,u) + B(s-t,u)^\top X_t,$$

where A and B satisfy the following ordinary differential equations (ODEs):

$$\dot{B} = (K^{\mathbb{Q}})^\top B + \frac{1}{2}\sum_{i=1}^{\mathrm{m}}[\Sigma^\top B]_i^2\beta_i + l_1\phi(B),$$

$$\dot{A} = (\Lambda^{\mathbb{Q}})^\top B + \frac{1}{2}\sum_{i=1}^{\mathrm{m}}[\Sigma^\top B]_i^2\alpha_i + l_0\phi(B),$$

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where $B(t) = u$, $A(t) = 0$, and for any $h \in \mathbb{C}^{\mathrm{N}}$,

$$\phi(h) = \int_{\mathbb{R}^{\mathrm{N}}} (e^{h^\top z} - 1 - h^\top z)\bar{\nu}^{\mathbb{Q}}(dz).$$

Under our risk neutral specification, we have

$$\mathbb{E}_t^{\mathbb{Q}}\Big\{ \int_t^{t+\tau} \sigma_s^2 ds + \int_t^{t+\tau} \int_{\mathbb{R}} j^2 \nu_s^{\mathbb{Q}}(dj) \Big\} =: \mathbb{E}_t^{\mathbb{Q}} \left\{ \int_t^{t+\tau} f^{\mathbb{Q}}(X_s) ds \right\},$$

where $f^{\mathbb{Q}}(X_s) = \Pi_0^{\mathbb{Q}} + \Pi_1^{\mathbb{Q}\top} X + X^\top \Pi_2 X + \exp\left\{\Pi_3 + \Pi_4{}^\top X\right\}$, $\Pi_0^{\mathbb{Q}} = \Pi_0 + l_0 \int_{\mathbb{R}} j^2 \bar{\nu}^{\mathbb{Q}}(dj)$, $\Pi_1^{\mathbb{Q}} = \Pi_1 + l_1 \int_{\mathbb{R}} j^2 \bar{\nu}^{\mathbb{Q}}(dj)$, and $\bar{\nu}^{\mathbb{Q}}(dj)$ is the marginal distribution of jumps in Y.

Denote the transition density of the process X as $p(X_s|s-t, X_t)$, and let $u = -iv$ in Ψ with $v \in \mathbb{R}^{\mathrm{N}}$, we have

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\Big(f^{\mathbb{Q}}(X_s)\Big|X_t = x\Big) &= \int_{\mathbb{R}^{\mathrm{N}}} f^{\mathbb{Q}}(x')p(x'|s-t, x)dx' \\ =&\frac{1}{(2\pi)^{\mathrm{N}}} \int_{\mathbb{R}^{\mathrm{N}}} \int_{\mathbb{R}^{\mathrm{N}}} f^{\mathbb{Q}}(x') e^{iv^\top x'} \Psi(s, t, -iv, x) dx' dv.\end{aligned}$$

We then utilize Fourier Transform of the tempered distributions to simplify the integral with respect to x'. Consider the quadratic part first. Note that

$$\int_{\mathbb{R}^{\mathrm{N}}} (\Pi_0^{\mathbb{Q}} + (\Pi_1^{\mathbb{Q}})^\top x' + x'^\top (\Pi_2) x') e^{iv^\top x'} dx' = (2\pi)^{\mathrm{N}} \Big(\Pi_0^{\mathbb{Q}} - i(\Pi_1^{\mathbb{Q}})^\top \nabla_v - \nabla_v (\Pi_2) \nabla_v\Big)\delta(v),$$

where $\delta(\cdot)$ is a Dirac delta that satisfies $\int_{\mathbb{R}^{\mathrm{N}}} \delta(v)dv = 1$, and $\int_{\mathbb{R}^{\mathrm{N}}} \delta(v)g(v)dv = g(0)$ for any test function g. Therefore, by direct calculations we obtain

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\Big(\Pi_0^{\mathbb{Q}} + (\Pi_1^{\mathbb{Q}})^\top X_s + X_s{}^\top (\Pi_2) X_s \Big| X_t = x\Big) &= \int_{\mathbb{R}^{\mathrm{N}}} \Big(\Pi_0^{\mathbb{Q}} - i(\Pi_1^{\mathbb{Q}})^\top \nabla_v - \nabla_v \Pi_2 \nabla_v\Big)\delta(v)\Psi(s, t, -iv, x)dv \\ &= \Pi_0^{\mathbb{Q}} + (\Pi_1^{\mathbb{Q}})^\top \nabla_u \Psi(s, t, u, x)\Big|_{u=0} + \nabla_u \Pi_2 \nabla_u \Psi(s, t, u, x)\Big|_{u=0}.\end{aligned}$$

For the exponential part, similarly we have

$$\int_{\mathbb{R}^{\mathrm{N}}} e^{\Pi_3 + (\Pi_4)^\top x'} e^{iv^\top x'} dx' = (2\pi)^{\mathrm{N}} e^{\Pi_3} \delta(v - i\Pi_4),$$

so that we can derive

$$\mathbb{E}^{\mathbb{Q}}\Big(e^{\Pi_3 + (\Pi_4)^\top X_s}\Big| X_t = x\Big) = \int_{\mathbb{R}^{\mathrm{N}}} e^{\Pi_3}\delta(v - i\Pi_4)\Psi(s, t, -iv, x)dv = e^{\Pi_3}\Psi(s, t, \Pi_4, x).$$

The pricing formula for variance swaps follows immediately. Note that we have applied properties of tempered distributions to simplify the calculations, all of which can be found in Kanwal (2004). ∎

B Invariant Transformations

Proof of Proposition 2. To prove the existence, we extend Dai and Singleton (2000) and Ahn et al. (2002) to provide invariant transformations of the general model. These transformations lead to alternative specifications without altering the price of variance swaps (or more generally, the likelihood of the observables). We summarize the state factors, Brownian motions, jumps, and parameter vectors in θ:

$$\theta = \Big(X_t, W_t^{\mathbb{Q}}, Z_t^{\mathbb{Q}}, \Lambda^{\mathbb{Q}}, K^{\mathbb{Q}}, \Sigma, \{\alpha_i, \beta_i\}_{1\le i\le \mathrm{m}}, \bar{\nu}^{\mathbb{Q}}(\cdot, dz), \Pi_0, \Pi_1, \Pi_2, \Pi_3, \Pi_4\Big).$$

There are 4 classes of admissible transformations that ensure the transformed process to follow (1), (2), (3), and (4) of the main text (with different set of parameters), while maintaining the same observable implications:

An Affine Transformation $\mathscr{T}_A$ refers to $\mathscr{T}_A X_t = \mathscr{V} + \mathscr{L} X_t$, where $\mathscr{V}$ is an $\mathrm{N}\times 1$ vector and $\mathscr{L}$ is an $\mathrm{N}\times\mathrm{N}$ nonsingular matrix. As a result, $\mathscr{T}_A\theta$ is defined below.

$$\mathscr{T}_A\theta = \begin{pmatrix} \mathscr{V} + \mathscr{L}X_t, W_t^{\mathbb{Q}}, \mathscr{L}Z_t^{\mathbb{Q}}, \mathscr{L}\Lambda - \mathscr{L}K\mathscr{L}^{-1}\mathscr{V}, \mathscr{L}K\mathscr{L}^{-1}, \mathscr{L}\Sigma, \\ \{\alpha_i - \beta_i^\top \mathscr{L}^{-1}\mathscr{V}, \mathscr{L}^{\top^{-1}}\beta_i\}_{1\le i\le \mathrm{m}}, \bar{\nu}^{\mathbb{Q}}(\mathscr{L}^{-1}(\cdot + \mathscr{V}), \mathscr{L}dz), \\ \Pi_0 - (\Pi_1)^\top \mathscr{L}^{-1}\mathscr{V} + \mathscr{V}^\top \mathscr{L}^{\top^{-1}}(\Pi_2)\mathscr{L}^{-1}\mathscr{V}, \mathscr{L}^{\top^{-1}}\Pi_1 - 2\mathscr{L}^{\top^{-1}}\Pi_2\mathscr{L}^{-1}\mathscr{V}, \mathscr{L}^{\top^{-1}}\Pi_2\mathscr{L}^{-1}, \\ \Pi_3 - (\Pi_4)^\top \mathscr{L}^{-1}\mathscr{V}, \mathscr{L}^{\top^{-1}}\Pi_4 \end{pmatrix}.$$

An Orthonormal Rotation $\mathscr{T}_O$ refers to an affine transformation on the Brownian factor $W_t^{\mathbb{Q}}$ such that $\mathscr{T}_O W_t^{\mathbb{Q}} = O W_t^{\mathbb{Q}}$, where O is an orthonormal matrix satisfying $O^\top O = OO^\top = I_{\mathrm{N}\times\mathrm{N}}$.

$$\mathscr{T}_O\theta = \Big(\ X_t, OW_t^{\mathbb{Q}}, Z_t^{\mathbb{Q}}, \Lambda^{\mathbb{Q}}, K^{\mathbb{Q}}, \Sigma O^\top, \{\alpha_i, \beta_i\}_{1\le i\le \mathrm{m}}, \bar{\nu}^{\mathbb{Q}}(\cdot, dz), \Pi_0, \Pi_1, \Pi_2, \Pi_3, \Pi_4\ \Big).$$

A Diffusion Rescaling $\mathscr{T}_D$ rescales the diagonal elements of S_t by a nonsingular diagonal matrix D in $\mathbb{R}^{\mathrm{N}\times\mathrm{N}}$. That is,

$$\mathscr{T}_D\theta = \Big(\ X_t, W_t^{\mathbb{Q}}, Z_t^{\mathbb{Q}}, \Lambda^{\mathbb{Q}}, K^{\mathbb{Q}}, \Sigma D^{-1}, \{D_{ii}^2\alpha_i, D_{ii}^2\beta_i\}_{1\le i\le \mathrm{m}}, \bar{\nu}^{\mathbb{Q}}(\cdot, dz), \Pi_0, \Pi_1, \Pi_2, \Pi_3, \Pi_4\ \Big).$$

A Permutation $\mathscr{T}_P$ alters the order of state variables, which has no observable effect.

Using these transformations, we can impose normalizations on the process to achieve its canonical representation. Following exactly the same procedure as in Appendix C of Dai and Singleton (2000), we normalize the parameters in the dynamics of (2), while leaving the parameters in (3) unrestricted (barring from their positivity constraints). Once we have transformed any model of (2) into its canonical form, no restrictions can be imposed on the parameters in (3) without affecting this canonical form, so that the procedure achieves the maximal model.

To show the uniqueness, by the existence result, it is equivalent to prove that canonical forms of different types are not observationally equivalent under these invariant transformations. This is obvious from Dai and Singleton (2000), because the number of positive factors remains unchanged under admissible transformations. ∎

C Extended Canonical Forms

Here we provide canonical forms that allow for pure jump volatility factors. Each model of this class is assigned to a family $\mathbb{A}_{\mathrm{m,j}}(\mathrm{N},\mathrm{J})$, in which N is the number of Brownian state variables, J is the number of pure jump factors, while m and j are the number of independent linear combinations of those state variables that are positive, respectively. In the absence of pure jump factors, we recycle the notation $\mathbb{A}_{\mathrm{m}}(\mathrm{N})$ in Dai and Singleton (2000), and provide the canonical forms in the main text.

For each m and j, we partition $X^\top = (X^\top_{\mathrm{m}\times 1}, X^\top_{j\times 1}, X^\top_{(\mathrm{N-m})\times 1}, X^\top_{(J-j)\times 1})^\top$. The canonical representation takes a special form of equation (3) in the main text, where for $m > 0$,

$$K^{\mathbb{Q}} = \begin{pmatrix} K^{\mathbb{Q}}_{\mathrm{m}\times\mathrm{m}} & K^{\mathbb{Q}}_{\mathrm{m}\times j} & 0_{m\times(\mathrm{N-m})} & 0_{m\times(J-j)} \\ K^{\mathbb{Q}}_{j\times\mathrm{m}} & K^{\mathbb{Q}}_{j\times j} & 0_{j\times(\mathrm{N-m})} & 0_{j\times(J-j)} \\ K^{\mathbb{Q}}_{(\mathrm{N-m})\times\mathrm{m}} & K^{\mathbb{Q}}_{(\mathrm{N-m})\times j} & K^{\mathbb{Q}}_{(\mathrm{N-m})\times(\mathrm{N-m})} & K^{\mathbb{Q}}_{(\mathrm{N-m})\times(J-j)} \\ K^{\mathbb{Q}}_{(J-j)\times\mathrm{m}} & K^{\mathbb{Q}}_{(J-j)\times j} & K^{\mathbb{Q}}_{(J-j)\times(\mathrm{N-m})} & K^{\mathbb{Q}}_{(J-j)\times(J-j)} \end{pmatrix},$$

and $K^{\mathbb{Q}}_{(\mathrm{N-m})\times(\mathrm{N-m})}$ and $K^{\mathbb{Q}}_{(J-j)\times(J-j)}$ is either the upper or lower triangle for $\mathrm{m} = 0$ or $j = 0$, respectively. In addition,

$$\Lambda^{\mathbb{Q}} = \begin{pmatrix} \Lambda^{\mathbb{Q}}_{\mathrm{m}\times 1} \\ 0_{j\times 1} \\ 0_{(\mathrm{N}+J-\mathrm{m}-j)\times 1} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1_{\mathrm{N}\times\mathrm{N}} & \\ & 0_{J\times J} \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0_{(\mathrm{m}+j)\times 1} \\ 1_{(\mathrm{N-m})\times 1} \\ 0_{(J-j)\times 1} \end{pmatrix},$$

$$\mathcal{B} = \begin{pmatrix} I_{\mathrm{m}\times\mathrm{m}} & 0_{\mathrm{m}\times j} & B_{\mathrm{m}\times(\mathrm{N-m})} & 0_{\mathrm{m}\times(J-j)} \\ B_{j\times m} & 0_{j\times j} & B_{j\times(\mathrm{N-m})} & 0_{j\times(J-j)} \\ 0_{(\mathrm{N-m})\times\mathrm{m}} & 0_{(\mathrm{N-m})\times j} & 0_{(\mathrm{N-m})\times(\mathrm{N-m})} & 0_{(\mathrm{N-m})\times(J-j)} \\ 0_{(J-j)\times\mathrm{m}} & 0_{(J-j)\times j} & 0_{(J-j)\times(\mathrm{N-m})} & 0_{(J-j)\times(J-j)} \end{pmatrix}, \quad l_1^{\mathbb{Q}} = \begin{pmatrix} l^{\mathbb{Q}}_{1,\mathrm{m}\times 1} \\ l^{\mathbb{Q}}_{1,j\times 1} \\ 0_{(\mathrm{N-m})\times 1} \\ 0_{(J-j)\times 1} \end{pmatrix},$$

with restrictions such that for $1 \le i \ne k \le \mathrm{m}$, $(\mathrm{m}+1) \le s \ne t \le \mathrm{m}+j$ and $1 \le j \le \mathrm{N}+J$,

$$K^{\mathbb{Q}}_{i,k} \ge 0, \quad K^{\mathbb{Q}}_{s,t} \ge 0, \quad \mathrm{Re}(\mathrm{Eigen}(\bar{K}^{\mathbb{Q}})) < 0, \quad \Lambda^{\mathbb{Q}}_i - l_0 \int_{\mathbb{R}} z_i \bar{\nu}^{\mathbb{Q}}(z)dz \ge \frac{1}{2}, \quad \Lambda^{\mathbb{Q}}_s \ge 0,$$
$$\mathcal{B}_{ij} \ge 0, \quad \mathcal{B}_{sj} \ge 0, \quad l^{\mathbb{Q}}_{1,i} \ge 0, \quad l^{\mathbb{Q}}_{1,s} = 1 \text{ or } 0, \quad l^{\mathbb{Q}}_0 \ge 0, \quad l^{\mathbb{Q}}_0 \Pi^{\mathrm{m}}_{i=1} l^{\mathbb{Q}}_{1,i} \Pi^{\mathrm{m}+j}_{s=\mathrm{m}+1} l^{\mathbb{Q}}_{1,s} \ne 0,$$
$$\bar{\nu}^{\mathbb{Q}}(\mathbb{R}^{\mathrm{m}+j}_- \times \mathbb{R}^{\mathrm{N}+J-\mathrm{m}-j}) = 0,$$

where $\bar{K}^{\mathbb{Q}}$ is of the same size of $K^{\mathbb{Q}}$, and

$$\bar{K}^{\mathbb{Q}}_{\mathrm{m}\times\mathrm{m}} = K^{\mathbb{Q}}_{\mathrm{m}\times\mathrm{m}} - \mathrm{Diag}\left(l_{1,i} \int_{\mathbb{R}} z_i \bar{\nu}^{\mathbb{Q}}(dz) \right)_{1\le i\le m},$$
$$\bar{K}^{\mathbb{Q}}_{j\times j} = K^{\mathbb{Q}}_{j\times j} - \mathrm{Diag}\left(l_{1,s} \int_{\mathbb{R}} z_s \bar{\nu}^{\mathbb{Q}}(dz) \right)_{\mathrm{m}+1\le s\le \mathrm{m}+j}.$$

D Summary of Two Factor Volatility Models

D.1 $\mathbb{A}_0(2)$ Model

The $\mathbb{A}_0(2)$ model specifies the dynamics of X as:

$$\begin{bmatrix} dX_{1t} \\ dX_{2t} \end{bmatrix} = \left(\begin{bmatrix} \kappa_{11}^{\mathbb{Q}} & 0 \\ \kappa_{21}^{\mathbb{Q}} & \kappa_{22}^{\mathbb{Q}} \end{bmatrix} \begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} \right) dt + \begin{bmatrix} dW_{1t}^{\mathbb{Q}} \\ dW_{2t}^{\mathbb{Q}} \end{bmatrix} + \begin{bmatrix} dZ_{1t}^{\mathbb{Q}} \\ dZ_{2t}^{\mathbb{Q}} \end{bmatrix}.$$

Jumps follow compound Poisson processes with independent jump sizes following double exponential distributions:

$$\text{size of } Z_{1t}^{\mathbb{Q}} \sim \begin{cases} & \exp(\beta_{1+}^{\mathbb{Q}}), \quad q_1 \\ - & \exp(\beta_{1-}^{\mathbb{Q}}), \quad 1 - q_1 \end{cases}, \quad \text{and}$$

$$\text{size of } Z_{2t}^{\mathbb{Q}} \sim \begin{cases} & \exp(\beta_{2+}^{\mathbb{Q}}), \quad q_2 \\ - & \exp(\beta_{2-}^{\mathbb{Q}}), \quad 1 - q_2 \end{cases}.$$

Their intensity is specified as l_0.

For this model, we specify the dynamics under $\mathbb{P}$ as

$$\begin{bmatrix} dX_{1t} \\ dX_{2t} \end{bmatrix} = \left(\begin{bmatrix} \lambda_1^{\mathbb{P}} \\ \lambda_2^{\mathbb{P}} \end{bmatrix} + \begin{bmatrix} \kappa_{11}^{\mathbb{P}} & 0 \\ \kappa_{21}^{\mathbb{P}} & \kappa_{22}^{\mathbb{P}} \end{bmatrix} \begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} \right) dt + \begin{bmatrix} dW_{1t}^{\mathbb{P}} \\ dW_{2t}^{\mathbb{P}} \end{bmatrix} + \begin{bmatrix} dZ_{1t}^{\mathbb{P}} \\ dZ_{2t}^{\mathbb{P}} \end{bmatrix},$$

where jumps in $Z_{1t}^{\mathbb{P}}$ and $Z_{2t}^{\mathbb{P}}$ are specified with the same mixture probabilities but in different sizes $\beta_{1,+/-}^{\mathbb{P}}$ and $\beta_{2,+/-}^{\mathbb{P}}$.

The parameter constraints in this model are given by:

$$\kappa_{11}^{\mathbb{Q}} < 0, \quad \kappa_{22}^{\mathbb{Q}} < 0, \quad l_0 \geq 0, \quad \kappa_{11}^{\mathbb{P}} < 0, \quad \kappa_{22}^{\mathbb{P}} < 0.$$

D.2 $\mathbb{A}_1(2)$ Model

Another model that incorporates negative jumps can be specified as

$$\begin{bmatrix} dX_{1t} \\ dX_{2t} \end{bmatrix} = \left(\begin{bmatrix} \lambda_1^{\mathbb{Q}} \\ 0 \end{bmatrix} + \begin{bmatrix} \kappa_{11}^{\mathbb{Q}} & 0 \\ \kappa_{21}^{\mathbb{Q}} & \kappa_{22}^{\mathbb{Q}} \end{bmatrix} \begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} \right) dt + \begin{bmatrix} \sqrt{X_{1t}} & 0 \\ 0 & \sqrt{1 + \beta_{21} X_{1t}} \end{bmatrix} \begin{bmatrix} dW_{1t}^{\mathbb{Q}} \\ dW_{2t}^{\mathbb{Q}} \end{bmatrix} + \begin{bmatrix} dZ_{1t}^{\mathbb{Q}} \\ dZ_{2t}^{\mathbb{Q}} \end{bmatrix},$$

where X_1 is a square-root factor, and X_2 is an Ornstein-Uhlenbeck factor. Jumps of X_1 and X_2 follow compound Poisson processes with independent jump sizes satisfying the exponential or double exponential distributions:

$$\text{size of } Z_{1t}^{\mathbb{Q}} \sim \exp(\beta_{1+}^{\mathbb{Q}}), \quad \text{and} \quad \text{size of } Z_{2t}^{\mathbb{Q}} \sim \begin{cases} & \exp(\beta_{2+}^{\mathbb{Q}}) \quad \text{with probability } q_2 \\ - & \exp(\beta_{2-}^{\mathbb{Q}}) \quad \text{with probability } 1 - q_2 \end{cases}.$$

Their intensity is specified as $l_0 + l_{11} X_{1t}$.

For this model, we specify the dynamics under the objective measure $\mathbb{P}$ as

$$\begin{bmatrix} dX_{1t} \\ dX_{2t} \end{bmatrix} = \left(\begin{bmatrix} \lambda_1^{\mathbb{P}} \\ \lambda_2^{\mathbb{P}} \end{bmatrix} + \begin{bmatrix} \kappa_{11}^{\mathbb{P}} & 0 \\ \kappa_{21}^{\mathbb{P}} & \kappa_{22}^{\mathbb{P}} \end{bmatrix} \begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} \right) dt + \begin{bmatrix} \sqrt{X_{1t}} & 0 \\ 0 & \sqrt{1+\beta_{21}X_{1t}} \end{bmatrix} \begin{bmatrix} dW_{1t}^{\mathbb{P}} \\ dW_{2t}^{\mathbb{P}} \end{bmatrix} + \begin{bmatrix} dZ_{1t}^{\mathbb{P}} \\ dZ_{2t}^{\mathbb{P}} \end{bmatrix}.$$

Jumps are of the same type with the same intensity and mixture probability but different sizes $\beta_{1+}^{\mathbb{P}}$, $\beta_{2+}^{\mathbb{P}}$, and $\beta_{2-}^{\mathbb{P}}$.

The parameter constraints in this model are given by:

$$\begin{aligned} &\kappa_{11}^{\mathbb{Q}} < l_{11}\beta_{1+}^{\mathbb{Q}}, \quad \kappa_{22}^{\mathbb{Q}} < 0, \quad \lambda_1^{\mathbb{Q}} - l_0\beta_{1+}^{\mathbb{Q}} \geq \frac{1}{2}, \quad \beta_{21} \geq 0, \quad l_0 \geq 0, \quad l_{11} \geq 0, \\ &\kappa_{11}^{\mathbb{P}} < l_{11}\beta_{1+}^{\mathbb{P}}, \quad \kappa_{22}^{\mathbb{P}} < 0, \quad \lambda_1^{\mathbb{P}} - l_0\beta_{1+}^{\mathbb{P}} \geq \frac{1}{2}. \end{aligned}$$

D.3 $\mathbb{A}_2(2)$ Model

The dynamics of the state variables in the $\mathbb{A}_2(2)$ model is specified as

$$\begin{bmatrix} dX_{1t} \\ dX_{2t} \end{bmatrix} = \left(\begin{bmatrix} \lambda_1^{\mathbb{Q}} \\ \lambda_2^{\mathbb{Q}} \end{bmatrix} + \begin{bmatrix} \kappa_{11}^{\mathbb{Q}} & \kappa_{12}^{\mathbb{Q}} \\ \kappa_{21}^{\mathbb{Q}} & \kappa_{22}^{\mathbb{Q}} \end{bmatrix} \begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} \right) dt + \begin{bmatrix} \sqrt{X_{1t}} & 0 \\ 0 & \sqrt{X_{2t}} \end{bmatrix} \begin{bmatrix} dW_{1t}^{\mathbb{Q}} \\ dW_{2t}^{\mathbb{Q}} \end{bmatrix} + \begin{bmatrix} dZ_{1t}^{\mathbb{Q}} \\ dZ_{2t}^{\mathbb{Q}} \end{bmatrix},$$

where jumps in Z_{1t} and Z_{2t} cannot be negative. The intensity of jumps is $l_0 + l_{11}X_{1t} + l_{12}X_{2t}$. The corresponding $\mathbb{P}$ measure dynamics is specified as:

$$\begin{bmatrix} dX_{1t} \\ dX_{2t} \end{bmatrix} = \left(\begin{bmatrix} \lambda_1^{\mathbb{P}} \\ \lambda_2^{\mathbb{P}} \end{bmatrix} + \begin{bmatrix} \kappa_{11}^{\mathbb{P}} & \kappa_{12}^{\mathbb{P}} \\ \kappa_{21}^{\mathbb{P}} & \kappa_{22}^{\mathbb{P}} \end{bmatrix} \begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} \right) dt + \begin{bmatrix} \sqrt{X_{1t}} & 0 \\ 0 & \sqrt{X_{2t}} \end{bmatrix} \begin{bmatrix} dW_{1t}^{\mathbb{P}} \\ dW_{2t}^{\mathbb{P}} \end{bmatrix} + \begin{bmatrix} dZ_{1t}^{\mathbb{P}} \\ dZ_{2t}^{\mathbb{P}} \end{bmatrix},$$

with exponentially distributed jumps and different mean parameters.

The parameter constraints in this model are given by:

$$\begin{aligned} &\mathrm{Re}\left(\mathrm{Eigen}\begin{pmatrix} \kappa_{11}^{\mathbb{Q}} - l_{11}\beta_{1+}^{\mathbb{Q}} & \kappa_{12}^{\mathbb{Q}} \\ \kappa_{21}^{\mathbb{Q}} & \kappa_{22}^{\mathbb{Q}} - l_{22}\beta_{2+}^{\mathbb{Q}} \end{pmatrix}\right) < 0, \quad \lambda_1^{\mathbb{Q}} - l_0\beta_{1+}^{\mathbb{Q}} \geq \frac{1}{2}, \quad \lambda_2^{\mathbb{Q}} - l_0\beta_{2+}^{\mathbb{Q}} \geq \frac{1}{2}, \\ &\mathrm{Re}\left(\mathrm{Eigen}\begin{pmatrix} \kappa_{11}^{\mathbb{P}} - l_{11}\beta_{1+}^{\mathbb{P}} & \kappa_{12}^{\mathbb{P}} \\ \kappa_{21}^{\mathbb{P}} & \kappa_{22}^{\mathbb{P}} - l_{22}\beta_{2+}^{\mathbb{P}} \end{pmatrix}\right) < 0, \quad \lambda_1^{\mathbb{P}} - l_0\beta_{1+}^{\mathbb{P}} \geq \frac{1}{2}, \quad \lambda_2^{\mathbb{P}} - l_0\beta_{2+}^{\mathbb{P}} \geq \frac{1}{2}, \\ &\kappa_{12}^{\mathbb{Q}} \geq 0, \quad \kappa_{21}^{\mathbb{Q}} \geq 0, \quad \kappa_{12}^{\mathbb{P}} \geq 0, \quad \kappa_{21}^{\mathbb{P}} \geq 0, \quad l_0 \geq 0, \quad l_{11} \geq 0, \quad l_{12} \geq 0. \end{aligned}$$

E Likelihood Inference in Detail

Below we give a more detailed description of the Gibbs blocks used in the posterior simulator. For the purpose of concreteness in this section we focus on the $\mathbb{A}_1(2)$ model, which contains one Ornstein-Uhlenbeck factor and one square-root factor.

E.1 Time Discretization and Joint Likelihood

A time discretization of the model with time interval Δ yields

$$y_i := Y_i - Y_{(i-1)} = \mu\Delta + \sigma_{i-1}\sqrt{\Delta}\left[\sqrt{1-\rho_1^2-\rho_2^2}\epsilon_{0i} + \rho_1\epsilon_{1i} + \rho_2\epsilon_{2i}\right] + j_i n_i, \tag{E.1}$$

$$X_{1i} - X_{1(i-1)} = \left[\lambda_1 + \kappa_{11}X_{1(i-1)}\right] + \sqrt{X_{1(i-1)}\Delta}\epsilon_{1i} + z_{1,i}n_i,$$

$$X_{2i} - X_{2(i-1)} = \left[\lambda_2 + \kappa_{21}X_{1(i-1)} + \kappa_{22}X_{2(i-1)}\right]\Delta + \sqrt{X_{2(i-1)}\Delta}\epsilon_{2i} + z_{2,i}n_i,$$

where n_i denotes the jump time indicator that takes the value one if there is a jump on that day, and ϵ_{0i}, ϵ_{1i}, and ϵ_{2i} are standard normal variates with zero correlations, j_i, $z_{1,i}$ and $z_{2,i}$ are Gaussian, Gamma, and mixture of Gammas, respectively. Note that $\mu = \mu^{\mathbb{P}} - l_0\mu_J^{\mathbb{P}}$, $\lambda_1 = \lambda_1^{\mathbb{P}} - l_0\beta_{1+}^{\mathbb{P}}$, $\lambda_2 = \lambda_2^{\mathbb{P}} - l_0(q_2\beta_{2+}^{\mathbb{P}} - (1-q_2)\beta_{2-}^{\mathbb{P}})$, $\kappa_{11} = \kappa_{11}^{\mathbb{P}} - l_{11}\beta_{1+}^{\mathbb{P}}$, $\kappa_{21} = \kappa_{21}^{\mathbb{P}} - l_{11}(q_2\beta_{2+}^{\mathbb{P}} - (1-q_2)\beta_{2-}^{\mathbb{P}})$, and $\kappa_{22} = \kappa_{22}^{\mathbb{P}} - l_{12}(q_2\beta_{2+}^{\mathbb{P}} - (1-q_2)\beta_{2-}^{\mathbb{P}})$.

The joint likelihood of the observables is then given by:

$$\mathcal{L}(Y, P\,|V, \Theta, \mathbb{A}_1(1)\,) = \prod_{i=1}^{T} p(y_i, X_i|X_{i-1}, j_i, z_i, n_i) \times p(j_i, z_i, n_i|X_{i-1}) \times p(P_i|X_i, \Theta),$$

which includes both the likelihood from the Euler discretization of the process and the likelihood of the variance swap rates. Eraker et al. (2003) show that discretization performs well with daily data. Alternatively, one could introduce a set of auxiliary data points in between of each par of sampled latent variables and integrate them out of the likelihood function by MCMC.

E.2 Jump Times and Sizes

In our application the jump indicator n_i is a binary random variable (taking on 0 or 1). To compute the Bernoulli probability, we use the conditional density of increments to volatility and returns to get that $\Pr(n_i = 1|V, \Theta, Y, P)$, which is equal to

$$\frac{p(y_i, X_i|X_{(i-1)}, j_i, z_{1,i}, z_{2,i}, n_i = 1, \Theta) \times \Pr(n_i = 1|X_{1(i-1)})}{\sum_{s=0,1} p(y_i, X_i|X_{(i-1)}, j_i, z_{1,i}, z_{2,i}, n_i = s, \Theta) \times \Pr(n_i = s|X_{1(i-1)})},$$

where $p(y_i, X_i|X_{(i-1)}, j_i, z_{1i}, z_{2i}, n_i = s, \Theta)$ is trivariate normal with mean and covariance matrix that can be easily obtained from (E.1) and $\Pr(n_i = 1|X_{1(i-1)}) = (l_0 + l_{11}X_{1(i-1)})\Delta$. Not surprisingly, the conditional posterior of jump times does not depend on the option prices directly since option prices do themselves not depend on the jump indicator.

To sample j_i, we note from (9) in the main text that $p(j_i|y_i, X_i, X_{(i-1)}, n_i = 1)$ is proportional to

$$p(y_i|X_i, X_{(i-1)}, j_i, n_i = 1) \times p(j_i|n_i = 1).$$

Completing the square in the previous expression we can easily obtain the mean and variance for the conditional posterior of j_i which is normal.

Analogous computations allow us to sample $z_{1,i}$ and $z_{2,i}$, which have a discrete scale mixture of truncated normals (TN) with a mixing variate that takes a positive (negative) value with mean $\mu^*_{k+,i}$ ($\mu^*_{k-,i}$) that can be easily obtained for $k = 1, 2$ by completing the squares. That is, if $s_{k,i} \in \{0, 1\}$, with $\Pr(s_{k,i} = 1 | y_i, X_i, \Theta) = q_k$, then

$$z_{k,i} = s_{k,i} \cdot \text{TN}(\mu^*_{k+,i}, \sigma^{*2}_{k,i}; z_{k,i} > 0) + (1 - s_{k,i}) \cdot \text{TN}(\mu^*_{k-,i}, \sigma^{*2}_{k,i}; z_{k,i} < 0),$$

where $\sigma^{*2}_{k,i}$ denotes the corresponding conditional posterior variance of the jump size in $z_{k,i}$.

Finally, when $n_i = 0$, the conditional posteriors of j_i, $z_{1,i}$ and $z_{2,i}$ are the priors implied by the model assumptions, as the data provide no information about them.

E.3 Latent Factors

The conditional posterior for latent factors is not known in closed-form. To sample from it, we collect terms in (9) where X_i is included, which is proportional to

$$\begin{aligned} &p(X_{i+1} | X_i, z_{1,i+1}, z_{2,i+1}, n_{i+1}) \times p(X_i | X_{i-1}, z_{1,i}, z_{2,i}, n_i) \\ &\times p(y_{i+1} | X_i, X_{i+1}, j_{i+1}, n_{i+1}) \times p(y_i | X_{i-1}, X_i, j_i, n_i) \times p(P_i | X_i, \Theta) \times p(n_{i+1} | X_i), \end{aligned}$$

where the first five densities are Gaussian and the last term is binomial. At the g-th iteration of the sampler, we then draw from its conditional posterior using a random-walk Metropolis algorithm with the Gaussian proposal density with mean and variance computed as in Proposition 2 of Eraker (2001) but taking into account the presence of jumps. The acceptance rate of this step is in the 20-30% range for all models.

E.4 Θ_{M} and Θ_{Π}

Conditional on jump sizes, jump times, spot variance, short-term variance level, and remaining parameter vectors, the posterior of Θ_{M} is proportional to (9). Since this conditional distribution is nonstandard, it is sampled using a Metropolis step with a normal source density centered at the current draw and covariance matrix proportional to the Hessian of $\mathcal{L}(Y, P | V, \Theta, M) \cdot \mathcal{H}(V | \Theta, M)$ at the peak of Θ_{M}. The Hessian was computed by concentrating the latent variables and remaining parameters on their posterior means from a preliminary run of the algorithm. An analogous but simpler procedure, since $\mathcal{H}(V | \Theta, M)$ does not appear in the conditional posterior, allows us to draw Θ_{Π}. The acceptance rate of this step is around 20% for all three models. The priors are relatively uninformative but still impose the relevant constraints.

E.5 Θ_{P} and Θ_{E}

A similar procedure to the one mentioned above can be used to sample Θ_{P}. In practice, however, since those parameters do not depend on variance swap rates once we condition on V, it is often the case that the conditional posterior distribution is available and therefore one can sample from it directly. The same comment applies to the variances of pricing errors as long as one chooses appropriately both the pricing error distributions and priors.

As for $\beta^{\mathbb{P}}_{1+}$, recall $z_{1,i} \sim Exponential(\beta^{\mathbb{P}}_{1+})$, so that conditional on $z_{1,i}$, and setting a conjugate prior for $\beta^{\mathbb{P}}_{1+}$, say $\pi_{\beta_{1+}}(\beta_{1+}) \sim InvGam(\delta_{\beta_{1+}1}, \delta_{\beta_{1+}2})$, we have that $\beta^{\mathbb{P}}_{1+}|... \sim invGam(\delta^*_{\beta_{1+}1}, \delta^*_{\beta_{1+}2})$ with $\delta^*_{\beta_{1+}1} = N_J + \delta_{\beta_{1+}1}$ and $\delta^*_{\beta_{1+}2} = \delta_{\beta_{1+}2} + \sum_{i=1}^{N_J} z_{1,i}$ and where $N_J = \sum_{i=1}^{T} n_i$. Similarly, we proceed with $\beta^{\mathbb{P}}_{2+}$ and $\beta^{\mathbb{P}}_{2-}$, but using the appropriate sample sizes $N_J^+ = \sum_{i=1}^{\mathrm{T}} n_i 1_{\{z_{2,i}>0\}}$ and $N_J^- = N_J - N_J^+$ and with $\pi_{\beta_{2+}}(\beta_{2+}) \sim invGam(\delta_{\beta_{2+}1}, \delta_{\beta_{2+}2})$ and $\pi_{\beta_{2-}}(\beta_{2-}) \sim invGam(\delta_{\beta_{2-}1}, \delta_{\beta_{2-}2})$ being the corresponding priors.

Conditional on X, $\beta^{\mathbb{P}}_{1+}$, $\beta^{\mathbb{P}}_{2+}$, $\beta^{\mathbb{P}}_{2-}$, q_2, Θ_{M} and jump times and sizes, using the jump adjusted processes $\tilde{Y}_i = Y_i - j_i n_i$, $\tilde{X}_{1i} = X_{1i} - z_{1,i} n_i$ and $\tilde{X}_{2i} = X_{2i} - z_{2,i} n_i$, we can sample $\mu^{\mathbb{P}}$, $\kappa^{\mathbb{P}}_{11}$, $\kappa^{\mathbb{P}}_{22}$, and $\kappa^{\mathbb{P}}_{21}$ using the standard normal multivariate regression model with known variance. In this context, prior information for those parameters can be easily introduced through Gaussian conjugate priors. Finally, we use a Metropolis step to compute the conditional posterior of ρ_1 and ρ_2, which is proportional to $p(y_i, X_i|X_{i-1}, j_i, z_i, n_i)$.

Article	Data	Period	Frequency	Model	VRP 1M	VRP 2M	VRP 1Y
Aït-Sahalia et al. (2014)	VS	1996 - 2010	Daily	AJD		[-7%, 0%]	[-10%, -0.5%]
Amengual (2008)	VS	1996 - 2007	Daily	AJD		[-5%, 0%]	[-7%, -0.5%]
Bollerslev et al. (2011)	SPO	1990 - 2003	Intraday	MF	[-20%, 5%]*		
Carr and Wu (2009)	SPO	1996 - 2003	Intraday	MF	-2.74% (3.63)**		
Corradi et al. (2013)	VIX	1950 - 2006	Monthly	AD	[-30%, 8%]*		
Fan et al. (2016)	VIX	2006 - 2011	Intraday	MF	[-30%, 20%]*		
Fusari and Gonzalez-Perez (2012)	SPO	1996 - 2010	Daily	Log-OU	[-20%, 3%]		[-23%, 1%]
Todorov (2010)	SPO	1990 - 2002	Intraday	MF	[-6%, 2%]		
Zhou (2009)	VIX	1990 - 2008	Intraday	MF	[-200%, 40%]		

Table A.1: Selection of Literature Reporting Variance Risk Premia Estimates

Note: This table collects estimates from papers that report either the point estimates or time series plots of variance or volatility risk premia. In the Data column, "VS" denotes variance swaps, "SPO" denotes S&P 500 options, and "VIX" is the CBOE volatility index. In the Model column, "AD" refers to affine diffusion, "AJD" denotes affine-jump diffusion, "MF" means model-free, and "Log-OU" is a log-affine process with two Ornstein-Uhlenbeck factors. The columns of VRP provide the approximate bounds of the estimated time series of risk premia, with 1M, 3M and 1Y referring to the 1-month, 2-month, and 1-year time-to-maturities. Most MF methodologies provide positive estimates of variance risk premia for certain periods of time, so that their upper bounds are positive.

* provides volatility risk premia.

** gives a point estimate with the standard error provided in the brackets.

	$\mathbb{A}_0(2)$				$\mathbb{A}_1(2)$				$\mathbb{A}_2(2)$			
Parameter	True	Bias	Stdev	HPD 95%	True	Bias	Stdev	HPD 95%	True	Bias	Stdev	HPD 95%
$\lambda_1^{\mathbb{Q}}$					6.000	0.000	0.006	[5.983, 6.009]	6.000	0.000	0.002	[5.996, 6.005]
$\lambda_2^{\mathbb{Q}}$									0.010	0.000	0.002	[0.006, 0.015]
$\kappa_{11}^{\mathbb{Q}}$	−4.500	0.001	0.016	[−4.537, −4.467]	−4.500	−0.008	0.013	[−4.536, −4.483]	−4.500	−0.002	0.008	[−4.521, −4.489]
$\kappa_{12}^{\mathbb{Q}}$									−0.012	−0.001	0.005	[−0.027, −0.006]
$\kappa_{21}^{\mathbb{Q}}$	1.800	0.002	0.013	[1.775, 1.828]	−0.480	−0.001	0.014	[−0.509, −0.453]	−0.200	0.001	0.008	[−0.214, −0.178]
$\kappa_{22}^{\mathbb{Q}}$	−0.200	0.000	0.006	[−0.213, −0.187]	−0.200	0.000	0.009	[−0.217, −0.178]	−0.250	−0.001	0.002	[−0.255, −0.247]
β_{21}					0.500	0.000	0.002	[0.497, 0.504]				
$\beta_{1+}^{\mathbb{Q}}$	0.210	0.001	0.004	[0.202, 0.219]	0.150	−0.001	0.005	[0.138, 0.158]	0.150	−0.001	0.002	[0.144, 0.154]
$\beta_{1-}^{\mathbb{Q}}$	0.040	0.002	0.004	[0.035, 0.049]								
$q_1^{\mathbb{Q}}$	0.400	0.000	0.004	[0.391, 0.407]								
$\beta_{2+}^{\mathbb{Q}}$	0.280	−0.001	0.003	[0.272, 0.285]	0.300	−0.002	0.005	[0.288, 0.309]	0.300	0.000	0.002	[0.296, 0.306]
$\beta_{2-}^{\mathbb{Q}}$	0.060	0.001	0.004	[0.052, 0.068]	0.100	−0.001	0.005	[0.087, 0.110]				
$q_2^{\mathbb{Q}}$	0.400	0.000	0.003	[0.394, 0.407]	0.450	0.000	0.005	[0.438, 0.459]				
l_0	10.000	0.000	0.003	[9.993, 10.006]	10.000	0.000	0.006	[9.988, 10.012]	10.000	0.000	0.003	[9.994, 10.004]
l_{11}					0.050	0.000	0.006	[0.038, 0.063]	0.040	0.000	0.002	[0.035, 0.046]
l_{12}									0.001	0.001	0.002	[0.000, 0.006]
$\mu_J^{\mathbb{Q}}$	−0.010	0.000	0.002	[−0.013, −0.006]	−0.010	0.001	0.002	[−0.013, −0.003]	−0.010	0.000	0.002	[−0.013, −0.005]
σ_J^2	0.020	0.000	0.001	[0.018, 0.022]	0.020	0.001	0.001	[0.019, 0.023]	0.020	0.000	0.001	[0.019, 0.022]
$\Pi_0 \times 10^4$	3.000	0.640	1.087	[1.928, 6.043]	20.000	0.152	1.218	[18.011, 23.126]	20.000	−0.047	1.218	[16.879, 22.296]
$\Pi_{11} \times 10^4$	12.000	−1.940	2.758	[1.873, 14.983]	−20.000	−0.165	2.293	[−26.405, −16.228]	2.000	0.630	1.787	[−0.661, 6.603]
$\Pi_{12} \times 10^4$	1.000	−0.048	1.397	[−1.490, 3.791]	2.000	−0.589	1.525	[−1.492, 4.982]	−12.000	1.031	1.567	[−13.481, −6.566]
$\Pi_{211} \times 10^4$	12.000	0.021	0.854	[9.541, 13.215]	10.000	−0.142	0.602	[8.503, 11.226]	12.000	−0.239	0.724	[10.016, 13.347]
$\Pi_{212} \times 10^4$	1.000	−0.161	0.395	[−0.086, 1.506]	0.000	0.225	0.457	[−0.551, 1.154]	−4.000	−0.197	0.587	[−5.402, −2.938]
$\Pi_{222} \times 10^4$	10.000	−0.023	0.263	[9.459, 10.482]	10.000	−0.002	0.312	[9.500, 10.850]	3.000	−0.005	0.377	[2.283, 4.049]
Π_3	−3.500	0.000	0.000	[−3.500, −3.500]	−3.850	0.000	0.000	[−3.850, −3.850]	−6.400	0.000	0.000	[−6.400, −6.400]
Π_{41}	1.500	0.000	0.000	[1.500, 1.500]	1.250	0.000	0.000	[1.250, 1.250]	1.250	0.000	0.000	[1.250, 1.250]
Π_{42}	0.500	0.000	0.000	[0.500, 0.500]	0.400	0.000	0.000	[0.400, 0.400]	0.400	0.000	0.000	[0.400, 0.400]

Table A.2: Simulation Results for Θ_{M} and Θ_{Π}

Note: This table provides a summary of a Monte Carlo simulation exercise with 100 replications for the two examples of two-factor volatility models that we introduce in Appendix **??**. We report true values, bias and standard deviations across the simulations for Θ_{M}, the parameters determining the dynamics of the latent factors under the risk-reutral measure, and Θ_{Π}, the parameters defining f.

	$\mathbb{A}_0(2)$				$\mathbb{A}_1(2)$				$\mathbb{A}_2(2)$			
Parameter	True	Bias	Stdev	HPD 95%	True	Bias	Stdev	HPD 95%	True	Bias	Stdev	HPD 95%
$\lambda_1^{\mathbb{P}}$	−1.750	−0.264	0.694	[−3.518, −0.903]	5.400	0.140	1.048	[3.698, 7.729]	7.500	0.184	1.524	[4.801, 10.751]
$\lambda_2^{\mathbb{P}}$	0.200	−0.315	0.938	[−2.203, 1.503]	−9.000	0.019	2.330	[−13.888, −4.639]	−3.500	0.839	2.078	[−6.635, 2.083]
$\kappa_{11}^{\mathbb{P}}$	−3.600	−0.260	0.789	[−5.653, −2.547]	−4.000	−0.087	0.728	[−5.611, −2.757]	−5.500	0.012	0.893	[−7.298, −3.801]
$\kappa_{12}^{\mathbb{P}}$									−0.250	−0.028	0.181	[−0.639, 0.078]
$\kappa_{21}^{\mathbb{P}}$	4.800	0.011	0.877	[2.965, 6.451]	4.000	−0.209	1.055	[1.574, 5.805]	4.500	−0.505	1.258	[1.447, 6.527]
$\kappa_{22}^{\mathbb{P}}$	−0.600	−0.093	0.222	[−1.185, −0.324]	−0.800	−0.089	0.303	[−1.558, −0.359]	−0.400	−0.054	0.267	[−1.024, 0.011]
$\beta_{1+}^{\mathbb{P}}$	0.200	−0.002	0.044	[0.119, 0.293]	0.100	0.005	0.018	[0.076, 0.145]	0.100	0.004	0.015	[0.077, 0.137]
$\beta_{1-}^{\mathbb{P}}$	0.180	0.009	0.041	[0.122, 0.284]								
$\beta_{2+}^{\mathbb{P}}$	0.200	0.008	0.044	[0.136, 0.309]	0.160	0.004	0.043	[0.084, 0.254]	0.120	−0.002	0.022	[0.080, 0.163]
$\beta_{2-}^{\mathbb{P}}$	0.200	0.005	0.032	[0.148, 0.274]	0.270	−0.004	0.051	[0.175, 0.373]				
$\mu_J^{\mathbb{P}}$	0.010	0.000	0.002	[0.006, 0.015]	0.010	0.000	0.003	[0.003, 0.016]	0.010	0.000	0.003	[0.005, 0.015]
ρ_1	−0.700	0.017	0.041	[−0.745, −0.578]	−0.600	−0.009	0.023	[−0.652, −0.564]	−0.450	−0.007	0.017	[−0.490, −0.424]
ρ_2	−0.450	0.018	0.040	[−0.510, −0.354]	−0.500	0.033	0.036	[−0.535, −0.399]	−0.750	0.002	0.012	[−0.772, −0.724]
$\mu_t^{\mathbb{P}}$	0.070	0.000	0.046	[−0.013, 0.169]	0.150	−0.010	0.077	[−0.008, 0.293]	0.120	−0.021	0.059	[−0.021, 0.206]
s_{2M}^2	0.150	0.001	0.007	[0.138, 0.165]	0.150	0.000	0.007	[0.138, 0.166]	0.150	0.001	0.007	[0.137, 0.166]
s_{3M}^2	0.010	0.001	0.000	[0.010, 0.012]	0.010	0.001	0.001	[0.010, 0.013]	0.010	0.002	0.001	[0.011, 0.013]
s_{6M}^2	0.050	0.000	0.002	[0.047, 0.053]	0.050	0.000	0.001	[0.047, 0.053]	0.050	0.000	0.002	[0.047, 0.054]
s_{9M}^2	0.060	0.000	0.002	[0.056, 0.065]	0.060	0.000	0.002	[0.056, 0.065]	0.060	0.000	0.002	[0.056, 0.065]
s_{1Y}^2	0.020	0.001	0.001	[0.019, 0.022]	0.020	0.001	0.001	[0.019, 0.022]	0.020	0.001	0.001	[0.019, 0.022]
s_{2Y}^2	0.050	0.001	0.002	[0.047, 0.055]	0.050	0.001	0.002	[0.047, 0.055]	0.050	0.001	0.002	[0.046, 0.055]

Table A.3: Simulation Results for Θ_{P} and Θ_{E}

Note: This table provides a summary of a Monte Carlo simulation exercise with 100 replications for the two examples of two-factor volatility models that we introduce in Appendix **??**. We report true values, bias and standard deviations across the simulations for the additional parameters that characterize the $\mathbb{P}$-dynamics, Θ_{P}, and the pricing error variances, summarized in Θ_{E}.

Indicator	Category	Frequency	Release Time
Unemployment Rate	Employment	Monthly	8:30 am First Friday of each month
ADP Employment Change	Employment	Monthly	8:15 am - Two days before Employment situation
Initial Jobless Claims	Employment	Weekly	8:30 am every Thursday
Personal Income	Consumer Spending and Confidence	Monthly	8:30 am 4 weeks after end of reported month
Personal Spending	Consumer Spending and Confidence	Monthly	8:30 am 4 weeks after end of reported month
Advance Retail Sales	Consumer Spending and Confidence	Monthly	8:30 am 2 weeks after end of reported month
Consumer Confidence	Consumer Spending and Confidence	Monthly	10:00 am - Last Tuesday of month being surveyed
GDP	National Output and Inventories	Quarterly	8:30 am - Final week of Jan Apr Jul Oct
Durable Goods Orders	National Output and Inventories	Monthly	8:30 am three to four weeks after the end of reporting month
ISM Manufacturing	National Output and Inventories	Monthly	10:00 am First Business day after reporting month
Chicago PMI	National Output and Inventories	Monthly	10:00 am First Business day of month being covered
Empire State Manufacturing	National Output and Inventories	Monthly	8:30 am around 15th of month being reported
Business Inventories	National Output and Inventories	Monthly	10:00 am released six weeks after the month ends
Production and Utilization	National Output and Inventories	Monthly	9:15 am released the 15th of the following month
New Residential Sales	Housing and Construction	Monthly	8:30 am released two to three weeks following month being covered
FOMC Meetings	Central Bank	Eight Times	2:15 pm of day of conclusion of FOMC meetings
Fed Chairman's Speeches	Central Bank	N.A.	N.A.
ECB Meetings	Central Bank	Monthly	N.A.
CPI	Prices, Productivity, Wages	Monthly	8:30 am second or third week following month being covered
PPI	Prices, Productivity, Wages	Monthly	8:30 am two or three weeks after month ends
Employment Cost Index	Prices, Productivity, Wages	Quarterly	8:30 am - Last Thursday of Jan Apr Jul Oct

Table A.4: Economic Indicators

Note: In this table, we report the details of the 21 macroeconomic news announcements or central bank events used in Section 4.1. All times are reported in Eastern Standard Time. Source: Bloomberg and the book by Baumohl (2010) on economic indicators.

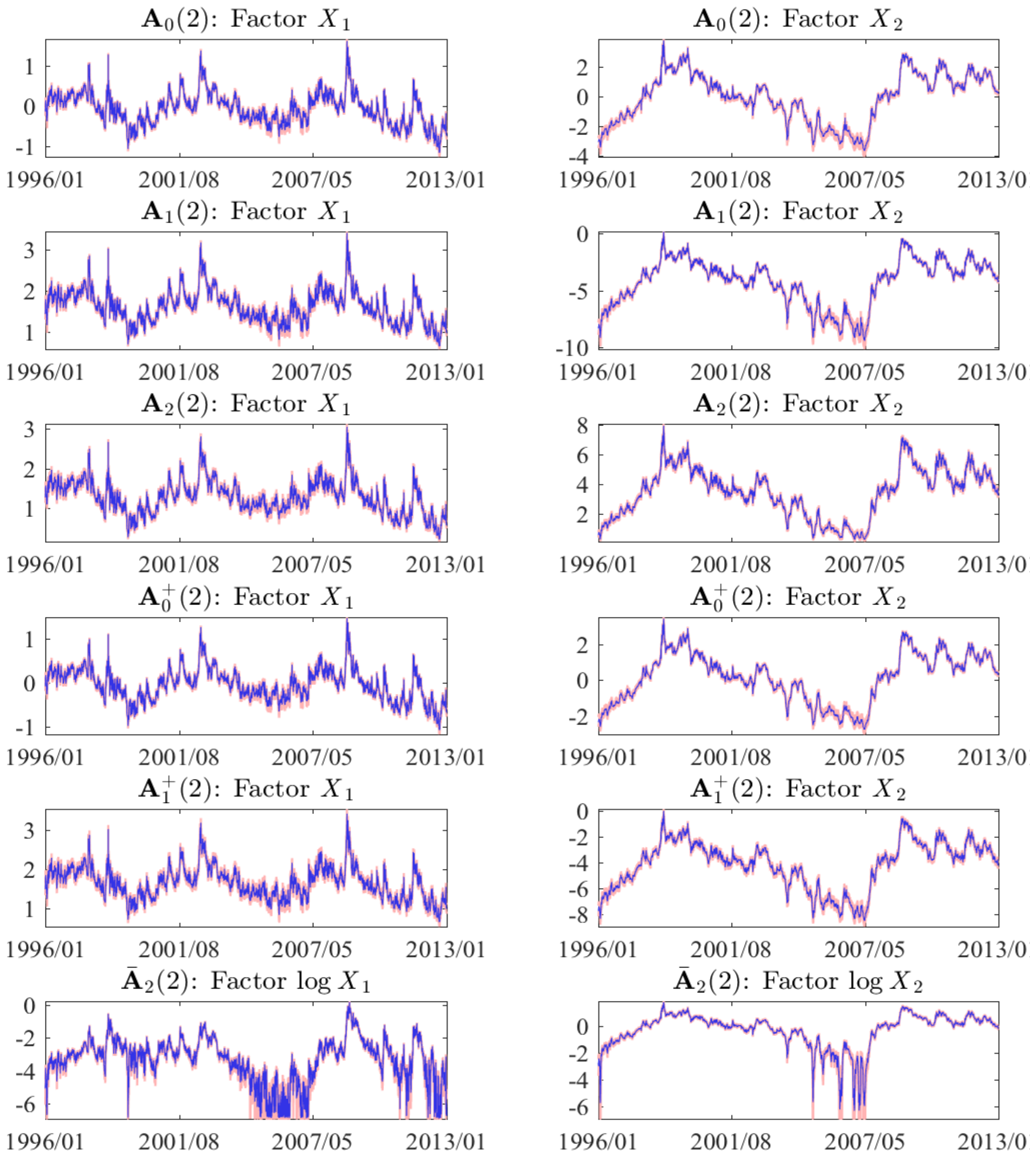

Figure A.1: Volatility Factors

Note: This figure reports the posterior means (in blue) of the two latent factor estimates for $\mathbb{A}_0(2)$, $\mathbb{A}_1(2)$, $\mathbb{A}_2(2)$, $\mathbb{A}_0^+(2)$, and $\mathbb{A}_1^+(2)$, as well as the logarithm of the factors for the $\bar{\mathbb{A}}_2(2)$ model. The red areas around the blue curves mark the 95%-credible sets. We use daily data on variance swaps from January 4, 1996 to January 11, 2013. The number of daily observations is 4,276, excluding weekends and holidays.

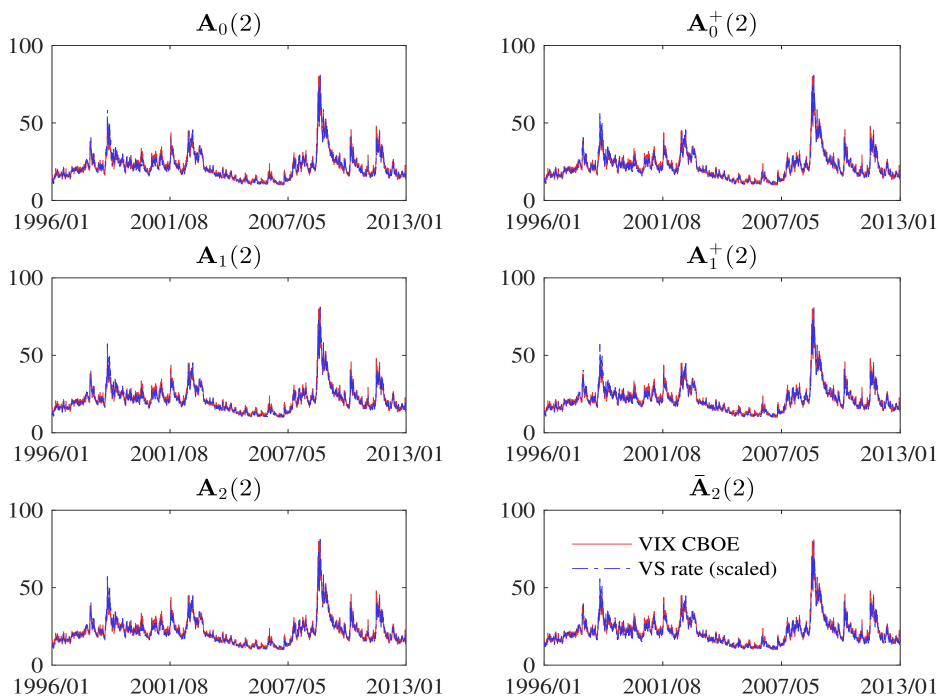

Figure A.2: Out of Sample Performance

Note: This figure compares the estimated 1-month variance swap rates with the VIX across models over the entire sample period. The red solid line denotes the VIX from the CBOE, whereas the blue dash-dotted line is calculated based on the $\mathbb{Q}$-parameters estimated from the variance swap rates with time-to-maturity of at least 2 months.

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